# Critical Conductance and Its Fluctuations at Integer Hall Plateau Transitions

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Under periodic boundary conditions in the transverse direction, we calculate the averaged zero-temperature two-terminal conductance $\langle G\rangle$ and its statistical fluctuation $\langle(\delta G)^{2n}\rangle$, for $n \leq 4$, at the critical point of integer quantum Hall plateau transitions. We find *universal* values for $\langle G\rangle = (0.58 \pm 0.03)e^2/h$, and $\langle(\delta G)^{2n}\rangle = (e^2/h)^{2n}A_{2n}$, where $A_{2,4,6,8} = 0.081 \pm 0.005,\ 0.013 \pm 0.003,\ 0.0026 \pm 0.005$, and $(8 \pm 2) \times 10^{-4}$, respectively. We also determine the leading finite size scaling corrections to these observables, and make comparisons with experiments. [S0031-9007(96)01658-4]

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For macroscopic, disordered, two-dimensional electronic systems at low temperatures, the metallic behavior is only observed near quantum phase transitions. Two examples are the superconductor to insulator transitions in amorphous thin films, and the transitions between quantum Hall plateaus [1]. In two spatial dimensions, the conductivity tensor $\sigma_{\mu\nu}$ measured in units of $e^2/h$ is dimensionless. Under generic conditions, $\langle\sigma_{\mu\nu}\rangle$ (the impurity averaged conductivities) are expected to be universal at the quantum critical points [2,3].

The universal conductance fluctuations of disordered mesoscopic metals has attracted tremendous experimental and theoretical interest in recent years [4]. The same question can be asked at the quantum critical points mentioned above. Thus "what is the statistical (over the impurity ensemble) properties of $\sigma_{\mu\nu}$ at quantum critical points of two-dimensional systems" is the concern of the present paper. In particular, we concentrate on the transitions between integer quantum Hall plateaus.

In an integer plateau transition, the electron conductivity tensor $(\sigma_{xx}, \sigma_{xy})$ changes from $(0, n)e^2/h$ to $(0, n \pm 1)e^2/h$. Concerning these transitions the following consensus is reached [1,5–8]. (a) When extrapolated to zero temperature and infinite sample size, the transitions are genuine continuous phase transitions with a *single* divergent length scale $\xi$—the quasiparticle localization length. (b) As the Fermi energy ($E_F$) moves toward a critical value $E_c$, $\xi \sim |E_F - E_c|^{-\nu}$, with $\nu \simeq 2.3$. (c) The characteristic energy scale $\sim 1/\xi$, thus (the dynamical exponent) $z = 1$.

Based on a Chern-Simons formalism, Kivelson, Lee, and Zhang (KLZ) asserted that $(\sigma_{xx}, \sigma_{xy})$ at the $(0, n)e^2/h \rightarrow (0, n + 1)e^2/h$ transition is $(\sigma_{xx}^c, \sigma_{xy}^c) = (\frac{1}{2}, n + \frac{1}{2})e^2h$ [3]. Huo, Hetzel, and Bhatt have numerically computed $\sigma_{xx}^c$ at the $(0,0) \rightarrow (0,1)e^2/h$ transition using the Kubo formula [9]. Their calculation assumes that the electronic states lie entirely in the lowest Landau level. They have considered several different forms of the impurity potential and concluded that $\sigma_{xx}^c = (0.55 \pm 0.05)e^2/h$ which is consistent with the result of KLZ.

Recently, the experimental four-terminal resistivity $\rho_{xx}$ has been studied systematically at the quantum Hall liquid to insulator transitions by Shahar *et al.* [10]. They concluded that the critical conductivity tensor is *universal*, and the value of $\rho_{xx}^c \approx h/e^2$ agrees with the prediction of KLZ. In addition, significant sample to sample fluctuations in the critical resistance have been observed [10]. Here we note that, in order to measure the true $\rho_{xx}^c$ (or $\sigma_{xx}^c$), the temperature has to be low enough so that corrections to scaling have faded away, yet it has to be high enough so as to avoid the effects of mesoscopic fluctuations [11]. More recently, the statistical fluctuation of the two-terminal conductance in the transition regime has been studied experimentally in mesoscopic samples by Cobden and Kogan who demonstrated the presence of large mesoscopic fluctuations in the conductance [12].

In this paper, we calculate the ensemble averaged *two-terminal* conductance $\langle G\rangle$ and, for the first time, its fluctuations $\langle(\delta G)^{2n}\rangle$ at the critical point of the integer quantum Hall plateau transitions. In addition, we study the finite-size and aspect ratio dependence of these quantities. The model we use is the Chalker-Coddington network model [13] and its extension [14], and with periodic boundary conditions applied in the transverse direction. These models have been shown to exhibit the correct critical properties of the integer plateau transitions [13–15]. Our findings are summarized as follows. The conductance of a $W \times L$ sample (where $W$ is the circumference and $L$ is the length of the cylinder) exhibits the following scaling behavior:

$$\langle G\rangle = \frac{e^2}{h}\mathcal{F}(L^{y_{\rm rel}}\Delta g_{\rm rel}, L^{-y_{\rm irr}}\Delta g_{\rm irr}, W/L). \qquad (1)$$

Here we have used the fact that $G/(e^2/h)$ is dimensionless. In the above, $\langle\cdots\rangle$ denotes the disorder ensemble averaging, $\Delta g_{\rm rel}$ and $\Delta g_{\rm irr}$ are the coupling constants conjugate to the relevant and the least-irrelevant operators, respectively. At the critical point, $\Delta g_{\rm rel} = 0$. We have performed finite-size scaling analysis of $\langle G\rangle_c$ on $L \times L$ samples to extract $\mathcal{F}(0,0,1)$ and $y_{\rm irr}$ to be $(0.58 \pm 0.03)$

 

and $0.55 \pm 0.15$, respectively. Thus the critical conductance $\langle G \rangle_c = (0.58 \pm 0.03)e^2/h$. We have also calculated the central moments $\langle (\delta G)^{2n} \rangle$ for $n \leq 4$ and have shown that they exhibit the following scaling behavior,

$$\langle \delta G^{2n} \rangle = \left(\frac{e^2}{h}\right)^{2n} \mathcal{F}_{2n}(L^{y_{\rm rel}} \Delta g_{\rm rel}, L^{-y_{\rm irr}} \Delta g_{\rm irr}, W/L). \quad (2)$$

We have determined the critical moments $\langle \delta G^{2n} \rangle_c = (e^2/h)^{2n} \mathcal{F}_{2n}(0,0,1)$. For $n = 1$, 2, 3, and 4, the values of $\mathcal{F}_{2n}$ are found to be $0.081 \pm 0.005$, $0.013 \pm 0.003$, $0.0026 \pm 0.005$, and $(8 \pm 2) \times 10^{-4}$, respectively. We verified the universality of these results and assert that all higher moments, and hence the *entire* distribution function $P(G)$, are universal at the transition. Since in the rest of this paper we *ignore* the electron-electron interaction, our conclusions are, at most, relevant to the *integer* plateau transitions. We also point out that it remains *unclear* at present about the relation between the two-terminal conductance $G$ and the $\sigma_{\mu\nu}$ derived from the Kubo formula in a closed system without contacts.

Let us consider *noninteracting* electrons in a strong magnetic field, and potentials that are smooth on the scale of the magnetic length. In this limit the plateau transition can be described by "quantum percolation" of semiclassical electron orbits in the Chalker-Coddington network model [13,14]. The latter consists of a square lattice of potential saddle points at which quantum tunneling between the edges of quantum Hall droplets take place [Fig. 1(a)]. Away from these vertices, the edge electrons propagate along the directed links with a fixed chirality set by the direction of the magnetic field. To account for the random areas of the Hall droplets, the edge electrons accumulate random Bohm-Aharonov phases while traversing the links of the network. At each saddle point, as shown in Fig. 1(b), there are two incoming and two outgoing edges states. The associated probability amplitudes are denoted by $Z_{1,\ldots,4}$. Because of current conservation, $|Z_1|^2 + |Z_2|^2 = |Z_3|^2 + |Z_4|^2$. With a choice of gauge, the quantum tunneling event at each node is then completely specified by a $2 \times 2$ transfer matrix,

$$\begin{pmatrix} Z_1 \\ Z_3 \end{pmatrix} = \begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{i\phi_2} \end{pmatrix} \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \times \begin{pmatrix} e^{i\phi_3} & 0 \\ 0 & e^{i\phi_4} \end{pmatrix} \begin{pmatrix} Z_4 \\ Z_2 \end{pmatrix}, \quad (3)$$

where $\phi_i \in [0, 2\pi)$ are random phases, and $\theta$ is a real number. Using Eq. (3) as the building block, we construct the $W \times W$ transfer matrix $T_i$ which propagates eigen wave functions on a cylinder of circumference $W$ in the $x$ direction. The total transfer matrix $T$ for the entire $W \times L$ system is given by the matrix product $T = \prod_{i=1}^{L} T_i$. For the details of transfer matrix calculations, readers are referred to the original work of Chalker and Coddington [13]. Here we merely emphasize a few important points.

In general, the tunneling parameter $\theta$ in Eq. (3) should be a random variable, depending on the local potential at the saddle point. However, in Ref. [14] it was shown that introducing randomness in $\theta$ did not change the universality class (i.e., did not change the localization length exponent $\nu = 2.33 \pm 0.03$) of the plateau transition. It turns out that the network model is situated at the critical point as long as $\theta$ is the same for all nodes [15]. Invariance under 90° rotation selects a particular one $[\theta = \theta_c = \ln(1 + \sqrt{2})]$ out of the family of critical models [13,15]. We shall present results for both isotropic systems ($\theta = \theta_c$) for which the two-terminal conductance along the $x$ direction ($G_{xx}$) and the $y$ direction ($G_{yy}$) are equal, and anisotropic systems (i.e., $\theta \neq \theta_c$) for which $G_{xx} \neq G_{yy}$. In the latter case we study the geometric mean $\sqrt{G_{xx}G_{yy}}$.

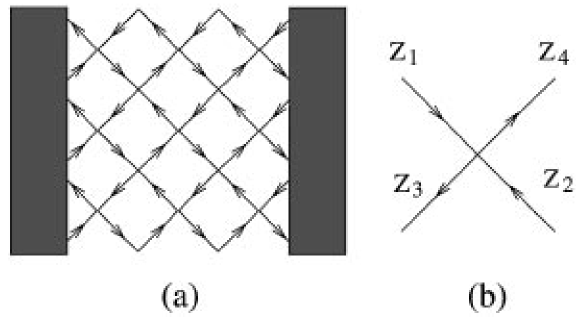


FIG. 1. (a) Schematics of the network model. The shaded areas correspond to the semi-infinite metallic leads. (b) The quantum tunneling at a single node of the network.

We next define the two-terminal conductance. As shown in Fig. 1(a), two semi-infinite conducting leads are connected to the $W \times L$ disordered network. It has been shown that linear response theory applied to the combined lead-sample-lead system gives the following multichannel two-terminal conductance [16,17]:

$$G = \frac{e^2}{h} \mathrm{Tr} t^\dagger t = \frac{e^2}{h} \sum_{i=1}^{W} \frac{1}{\cosh^2(\gamma_i L)}. \quad (4)$$

In Eq. (4), $t$ is the transmission matrix through the disordered region, and $\gamma_i$ is the $i$th of the $W$ Lyapunov exponents of the Hermitian transfer matrix product $T^\dagger T$ [18]. We note that the validity of Eq. (4) in the absence of time reversal symmetry has been shown in detail by Baranger and Stone for finite magnetic fields [17].

We now present the results. We have studied $W \times L$ systems with periodic boundary conditions in the transverse direction for $L = 8$, 16, 32, 48, 64, and 96, and for aspect ratio $W/L = 1/4$, $1/3$, $1/2$, 1, 2, and 4. The disorder ensemble consists of at least 3000 samples for each $(W, L)$. The distribution function $P(G)$ is very broad, and the most probable value of the critical conductance $G_{c,\rm typical}$ is, although close to $0.5e^2/h$, not sharply defined. For isotropic systems (i.e., $\theta = \theta_c$) the averaged critical conductance $\langle G(L) \rangle_c$ for $L \times L$ samples is shown in Fig. 2. To extract the asymptotic value $\langle G \rangle_c$

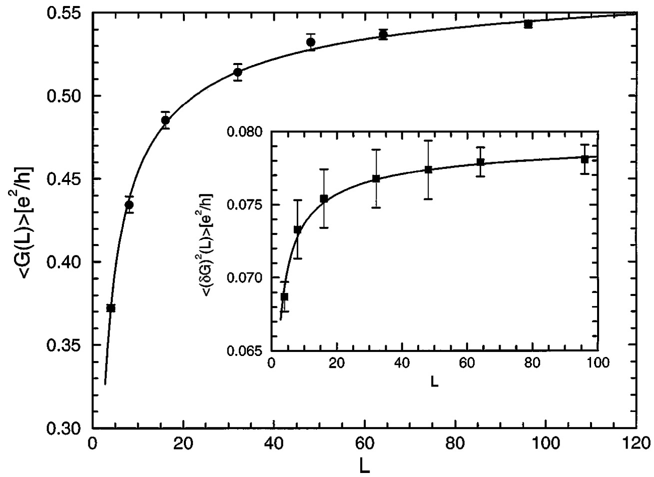


FIG. 2. The finite-size dependence of the critical conductance at the isotropic critical point. The solid line is the fit to the scaling form in Eq. (5) in the text with $\langle G\rangle_c = (0.58 \pm 0.03)e^2/h$ and $y_{\rm irr} = 0.55 \pm 0.15$. The inset shows the behavior of the conductance fluctuations, which gives $\langle(\delta G)^2\rangle_c = (0.081 \pm 0.005)\,(e^2/h)^2$ and the same renormalization group dimension $y_{\rm irr}$.

and the exponent $y_{\rm irr}$ at the critical point, we expand the scaling function in Eq. (1) according to $\mathcal{F}(0,x,1) \approx \mathcal{F}(0,0,1) + \mathcal{F}'(0,0,1)x$ for small $x$. Thus, for large system size $L$, we should have

$$\langle G(L)\rangle_c = \langle G\rangle_c + \mathcal{F}'(0,0,1)\Delta g_{\rm irr}L^{-y_{\rm irr}}, \qquad (5)$$

where $\langle G\rangle_c = \mathcal{F}(0,0,1)e^2/h$. From the data shown in Fig. 2, we obtain $\mathcal{F}(0,0,1) = 0.58 \pm 0.03$ and $y_{\rm irr} = 0.55 \pm 0.15$. We have also studied the aspect-ratio $(W/L)$ dependence of $\langle G\rangle_c$. In the context of Eq. (1) we find

$$\mathcal{F}(0,0,W/L) = c_1 e^{-c_2 L/W}(W/L)\,, \qquad (6)$$

with $c_1 = 0.72 \pm 0.03$ and $c_2 = 0.22 \pm 0.02$. The exponential factor is a precursor to the conductance behavior in a quasi-one-dimensional system with $L \gg W$.

In order to verify the universality of $\langle G\rangle_c$ and to check that the $-y_{\rm irr}$ is indeed the renormalization group dimension of the least-irrelevant operator, we have studied network models with various forms of node parameter disorder corresponding to random distributions of $\theta$ values with the same median $\theta_c$. The results are consistent within the error bars with those obtained above. This further supports the conclusion that $\theta$ disorder is an irrelevant perturbation at the critical point. In addition, we have also considered anisotropic critical models for which $\theta \neq \theta_c$. In this case, using the procedures described above, we have calculated the conductances $\langle G_{xx}\rangle$ and $\langle G_{yy}\rangle$ (periodic boundary conditions are always imposed in the transverse directions). The results are summarized in Table I. Although the critical values of $\langle G_{xx}\rangle_c$ and $\langle G_{yy}\rangle_c$ depend on the amount of anisotropy, their geometric mean $\sqrt{G_{xx}G_{yy}}$ does not. The latter stays close to the value of $\langle G\rangle_c$ obtained for the isotropic system. Thus we conclude that there exists a *universal* critical two-terminal conductance $\langle G\rangle_c = (0.58 \pm 0.03)e^2/h$.

The fact that our value for $\langle G\rangle_c$ deviates from $\sigma_{xx}^c = \frac{1}{2}(e^2/h)$ asserted by KLZ is, as far as we can see, real. However, as was pointed out at the beginning of this paper, it is not clear how the two-terminal conductance relates to the conductivity tensor derived from the Kubo formula in systems without contacts.

We now turn to the central moments of the critical conductance described by the scaling forms given in Eq. (2). In the inset of Fig. 2 we plot $\mathcal{F}_2 = \langle(\delta G)^2\rangle_c/(e^2/h)^2$ for the isotropic network (i.e., $\theta = \theta_c$) as a function of the system size for $L \times L$ samples. Following a similar finite-size scaling analysis as given in Eq. (5), we extract $\mathcal{F}_2(0,0,1) = 0.081 \pm 0.005$ and the same $y_{\rm irr} = 0.55 \pm 0.15$ to be the renormalization group dimension of the leading irrelevant operator. The dependence of $\langle(\delta G)^2\rangle_c$ on the degree of anisotropy (i.e., $\theta \neq \theta_c$) is shown in Table I. Thus we conclude $\langle(\delta G)^2\rangle_c = (0.081 \pm 0.005)\,(e^2/h)^2$. Repeating the procedure for the fourth and the sixth moments gives the results $\langle(\delta G)^4\rangle_c = (0.013 \pm 0.003)\,(e^2/h)^4$, $\langle(\delta G)^6\rangle_c = (0.0026 \pm 0.005)\,(e^2/h)^6$, and with less accuracy $\langle(\delta G)^8\rangle_c = (8 \pm 2) \times 10^{-4}(e^2/h)^8$. In Fig. 3, we present the data obtained for even-integer, as well as odd-integer and noninteger, central moments. The $n$th order moments interpolate between $\langle(\delta G)^n\rangle_c = a v^n e^{un^2}$ at small $n$ and $\langle(\delta G)^n\rangle_c = bn^{-\beta}$ at large $n$, where $(a, v, u, b, \beta) \approx (0.80, 0.28, 0054, 4.65, 4.18)$ are all universal constants. The small-$n$ behavior shows that *the critical conductance obeys a log-normal distribution.* On the other hand, the large-$n$ behavior, indicative of a broad distribution, is purely empirical. The latter should be contrasted to the behavior in mesoscopic disordered

TABLE I. Ensemble-averaged conductances and conductance fluctuations at the critical points of the isotropic ($\theta = \theta_c$) and the anisotropic ($\theta \neq \theta_c$) models. The units are $e^2/h$ for $G$ and $(e^2/h)^2$ for $(\delta G)^2$. Error estimates in the last digits are given in parenthesis.

| $\theta$ | $G_{xx}$ | $G_{yy}$ | $\sqrt{G_{xx}G_{yy}}$ | $(\delta G_{xx})^2$ | $(\delta G_{yy})^2$ | $\sqrt{(\delta G_{xx})^2(\delta G_{yy})^2}$ |
|---|---|---|---|---|---|---|
| $\theta_c$ | 0.58(3) | 0.58(3) | 0.58(3) | 0.081(5) | 0.081(5) | 0.081(5) |
| 0.84 | 0.64(3) | 0.53(3) | 0.58(3) | 0.081(5) | 0.084(5) | 0.082(5) |
| 0.80 | 0.75(3) | 0.47(3) | 0.59(3) | 0.080(5) | 0.086(3) | 0.082(4) |
| 0.75 | 0.88(3) | 0.43(3) | 0.62(4) | 0.075(5) | 0.083(5) | 0.079(5) |
| 0.70 | 1.04(3) | 0.30(3) | 0.56(4) | 0.079(6) | 0.079(5) | 0.079(6) |

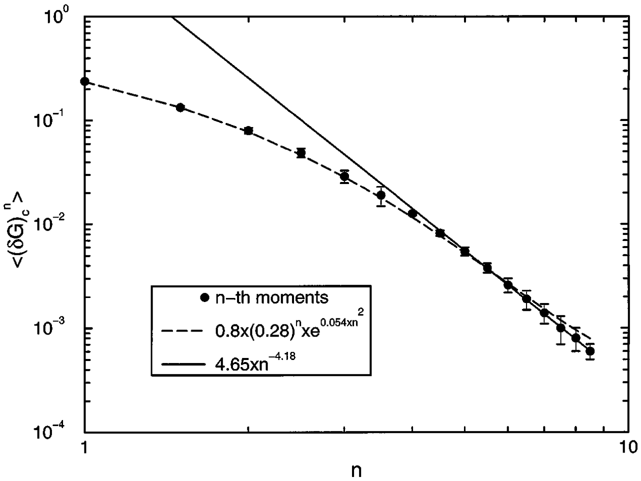


FIG. 3. The behavior of the $n$th order central moments of the conductance distribution which interpolates between the exponential and the power-law behaviors (dashed lines) in the small and large $n$ limits as described in the text.

metals, where the $(2 + e)$ expansion in the diffusive regime predicts nonuniversal moments for large $n$ [19].

We now compare our results with experimental findings. This comparison must be made under the *disclaimer* that, so far, the understanding of the effects of Coulomb interaction on the critical properties of the integer plateau transition is still incomplete [20]. First, we *assume* that the two-terminal conductance $\langle G \rangle_c$ is the four-probe $\sigma_{xx}^c$ measured experimentally. Second, from general particle-hole symmetry argument [3], one expects $\sigma_{xy}^c = 0.5e^2/h$. With our result $\langle G \rangle_c \approx 0.58e^2/h$, this implies $\rho_{xx}^c = 0.99h/e^2$. This value agrees with the experimental finding of Shahar *et al.* [10]. More recently, Cobden and Kogan have measured the two-terminal conductance near the integer transitions [12]. Since the latter was carried out on mesoscopic samples outside the asymptotic scaling regime, direct comparison with our results would be difficult to justify. Nevertheless, their data provide an estimated value $\langle (\delta G)^2 \rangle \approx 0.05(e^2/h)^2$, which is in reasonable agreement with our findings.

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