Anyons and Chiral Solitons on a Line

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We show that excitations in a recently proposed gauge theory for anyons on a line in fact do not obey anomalous statistics. On the other hand, the theory supports novel chiral solitons. Also we construct a field-theoretic description of lineal anyons, but gauge fields play no role. [S0031-9007(96)01685-7]

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A (nonrelativistic) field theoretic description for the quantum mechanics of planar particles with unconventional statistics—anyons—makes use of a Chern-Simons gauge field in a second quantized formalism. Recently there has appeared in these pages an article offering a similar description for particles on a line [1]. However, the claimed results are incorrect; apparently inattention to signs has led to error.

In this Letter, we analyze the model of Ref. [1]. We present an alternative nonrelativistic field theory, which succeeds in describing lineal anyons, but is not a gauge theory. Finally we show that the model of Ref. [1], though failing to achieve its announced goal, possesses an interesting and novel soliton structure.

(A) A nonrelativistic gauge field theory that leads to planar anyons is the nonlinear Schrödinger equation, gauged by a Chern-Simons field and governed by the Lagrange density

$$\begin{aligned}\mathcal{L}_{(2+1)} &= \frac{1}{4\kappa}\epsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma} + i\hbar\Psi^*(\partial_t + iA_0)\Psi \\ &\quad - \frac{\hbar^2}{2m}\sum_{i=1}^{2}|(\partial_i + iA_i)\Psi|^2 - V(\rho), \qquad (1)\\ \rho &\equiv \Psi^*\Psi .\end{aligned}$$

Here Ψ is the Schrödinger quantum field, giving rise to charged bosonic particles after (second) quantization. A_μ possesses no propagating degrees of freedom; it can be eliminated leaving a statistical Aharanov-Bohm interaction between the particles. V describes possible nonlinear self-interactions, e.g., $V(\rho) \propto \rho^2$ for the cubic Schrödinger equation.

When analyzing the lineal problem, it is natural to consider a dimensional reduction of (1), by suppressing dependence on the second spatial coordinate and renaming A_2 as $(mc/\hbar^2)B$. In this way one is led to a B-F gauge theory, described by the Lagrange density

$$\begin{aligned}\mathcal{L}_{(1+1)} &= \frac{1}{2\kappa}B\epsilon^{\mu\nu}F_{\mu\nu} + i\hbar\Psi^*(\partial_i + iA_0)\Psi \\ &\quad - \frac{\hbar^2}{2m}|(\partial_x + iA_x)\Psi|^2 - \frac{mc^2}{2\hbar^2}B^2\rho - V(\rho).\end{aligned} \tag{2}$$

Here $\kappa \equiv (\hbar^2/mc)\overline{\kappa}$ is dimensionless. (We retain physical constants: c is the light velocity, which plays no role in the following.) Eliminating the B and A_μ fields decouples them completely, in the sense that the phase of Ψ may be adjusted so that the interactions of the Ψ field are solely determined by V, and particle statistics remain unaffected [2].

[We observe here an interesting pattern: dimensional reduction of $\mathcal{L}_{(2+1)}$ in (1) and $V \propto \rho^2$, with respect to *space*, results in a completely integrable system on $(1 + 1)$-dimensional space-time: the nonlinear, cubic Schrödinger equation. On the other hand, reduction with respect to *time* results in a completely integrable system in two spatial dimensions (provided the cubic nonlinearity is of definite strength): the Liouville equation [3].]

In order to make the vector potential A_μ and the B field dynamically active, thereby allowing the Ψ particles to interact even in the absence of V, we include a kinetic term for B, which could be taken in the Klein-Gordon form. However, we prefer a simpler expression that describes "chiral" Bose fields, propagating in only one direction, whose Lagrangian density is proportional to $\pm\dot{B}B' - vB'B'$ [4] (dot/prime indicate differentiation with respect to time/space). Here v is a velocity and the consequent equations of motion for this kinetic term (without further interaction) are solved by $B = B(x \pm vt)$ (with suitable boundary conditions at spatial infinity), describing propagation with velocity $\mp v$. Note that $\dot{B}B'$ is not invariant against a Galileo transformation, which is a symmetry of $\mathcal{L}_{(1+1)}$ and of $B'B'$: performing a Galileo boost on $\dot{B}B'$ with velocity $\tilde{v}$ gives rise to $\tilde{v}B'B'$, effectively boosting the v parameter by $\tilde{v}$. Consequently one can drop the $vB'B'$ contribution to the kinetic B Lagrangian, thereby selecting to work in a global "rest frame." Boosting a solution in this rest frame then produces a solution to the theory with a $B'B'$ term.

In view of the above, we choose the B-kinetic Lagrange density to be

$$\mathcal{L}_B = \pm\frac{1}{\hbar}\dot{B}B' \tag{3}$$

and the total Lagrange density is

$$\mathcal{L} = \mathcal{L}_B + \mathcal{L}_{(1+1)} . \tag{4}$$

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It is still possible to remove the A_μ and B fields by a Hamiltonian reduction as described in Ref. [5], and by phase redefinition of Ψ. Once this is done, one is left with the Lagrangian

$$L = \int dx\, i\hbar \Psi^* \dot{\Psi} - H\,, \tag{5}$$

$$H = \frac{\hbar^2}{2m} \int dx\, \Pi^* \Pi\,, \tag{6}$$

$$\Pi \equiv (\partial_x \pm i\kappa^2 \rho)\Psi\,. \tag{7}$$

For simplicity, V has been omitted.

Quantization is straightforward: Ψ and Ψ^* are conjugates; when the Hamiltonian is taken in the form (6), which is Hermitian but *not* normal ordered, one derives the following Schrödinger equation for the two-body wave function $\psi(t;x_1,x_2)$:

$$\psi(t;x_1,x_2) \equiv \frac{1}{\sqrt{2}} \langle 0| \Psi(t,x_1)\Psi(t,x_2)\,|2\rangle\,, \tag{8a}$$

$$i\hbar\partial_t\psi = \frac{\hbar^2}{2m}\left\{\left[\frac{1}{i}\,\partial_{x_1} \pm \kappa^2\delta(x_1 - x_2)\right]^2 + \left[\frac{1}{i}\,\partial_{x_2} \pm \kappa^2\delta(x_2 - x_1)\right]^2\right\}\psi\,. \tag{8b}$$

This is the theory presented in Ref. [1]. It is in the subsequent analysis that the author goes wrong: He claims that the interaction with the "vector potential" $\pm\hbar\kappa^2\delta(x_1 - x_2)$ can be removed by redefining the wave function ψ with a step-function phase $\psi = \tilde{\psi} e^{\mp i\kappa^2\theta(x_1 - x_2)}$. But this is incorrect: A phase redefinition can remove the potential from the "1" kinetic term or from the "2" kinetic term, but not from both. Indeed removing the δ function in one term inserts it with the *same* sign in the other term. This is as it should be: We have already remarked that the theory is not Galileo invariant. More specifically, a two-body vector potential is Galileo invariant only if it is an odd function [6], while the δ function in (8) is even. Certainly one cannot transform a Galileo-noninvariant equation to a noninteracting, Galileo-invariant one.

We may solve Eq. (8). The theory is time- and space-translation invariant, so one can separate the time and center-of-mass coordinates

$$\psi(t;x_1,x_2) = e^{-i(Et/\hbar)} e^{i(P/\hbar)(x_1+x_2)/2} u(x_1 - x_2)\,. \tag{9}$$

The wave function for relative motion satisfies

$$\frac{1}{m}\left\{-\hbar^2\partial_x^2 + \left[\frac{P}{2} \pm \hbar\kappa^2\delta(x)\right]^2\right\}u(x) = Eu(x)\,. \tag{10}$$

The presence of the total momentum P vividly demonstrates the absence of Galileo invariance. Explicit solution requires regulating the δ function, to give meaning to $\delta(x)\delta(x)$. Once this is done, one finds a free odd solution

$$u_-(x) = \sin kx\,, \qquad \hbar^2k^2 = mE - P^2/4 \geq 0\,, \tag{11a}$$

while the even solution reads

$$u_+(x) = \sin k|x|\,; \tag{11b}$$

i.e., there is total reflection with π phase shift, and no transmission. Evidently the odd solution sees no potential, while the even one moves in an effective potential

$$V_{\text{effective}}(x) = \frac{P^2}{4m} + \frac{2\hbar^2}{m}\frac{\delta(x)}{|x|}\,. \tag{12}$$

Note that the coupling strength κ has disappeared. Alternatively, we may recognize the wave functions (11a) and (11b) as solutions to the Schrödinger equation with a δ-function potential, in the limit that its strength becomes infinite. Both from (10) and (12) we see that the interaction is repulsive and there are no bound states; see, however, Sec. C below.

(B) The gauge theoretic model of Ref. [1] fails to describe particles with arbitrary statistics. We now give a quantum field theory that does the job, but it is not a gauge theory.

The quantum mechanical description is well known [7]: It makes use of the scale-invariant $1/x^2$ potential, which may also be represented with the help of an exchange operator R in a "covariant" derivative [8]. The two-body equation reads

$$i\hbar\partial_t\psi(t;x_1,x_2) = \frac{\hbar^2}{2m}\left[\left(\frac{1}{i}\,\partial_{x_1} + \frac{i\nu}{x_1 - x_2}R\right)^2 + \left(\frac{1}{i}\,\partial_{x_2} + \frac{i\nu}{x_2 - x_1}R\right)^2\right]\psi(t;x_1,x_2)\,, \tag{13}$$

where $Rf(x_1,x_2) = f(x_2,x_1)$. Hence acting on even wave functions (13) gives

$$i\hbar\partial_t\psi = \frac{\hbar^2}{2m}\left[-\partial_{x_1}^2 - \partial_{x_2}^2 + \frac{2\nu(\nu - 1)}{(x_1 - x_2)^2}\right]\psi\,. \tag{14}$$

(A similar expression is gotten with odd wave functions.) Note that the covariant derivatives in (13) commute, but the interaction cannot be removed, since it is equivalent to (14).

For a field-theoretical description of the many-body problem, we use a Lagrangian and Hamiltonian as in (5), (6) with

$$\Pi(t,x) = \left[\partial_x + \nu \int dy\, \frac{1}{x - y}\,\rho(t,y)\right]\Psi(t,x)\,. \tag{15}$$

This is *not* a gauge-covariant derivative; being real, $\int dy\, \rho(t,y)/(x - y)$ is *not* a gauge connection. Note the Hamiltonian (6) is *not* normal ordered, but one can reorder H: with Π given by (15), one finds

$$H = \frac{\hbar^2}{2m}\left(\int dx\, \Psi^{*\prime}\Psi' + \nu(\nu - 1) \times \int dx\, dy\, \frac{:\rho(t,x)\rho(t,y):}{(x - y)^2}\right)\,. \tag{16}$$

This is precisely the field theoretic description of (14). [The term involving six Ψ fields vanishes by symmetry, provided no attention is paid to singularities at coincident points $x_i = x_j$. If a principal value prescription is posited, one would use the identity $P\frac{1}{x-y}P\frac{1}{x-z} + 2$ permutations $= \pi^2\delta(x-y)\delta(x-z)$ and H would acquire the addition $\frac{\hbar^2}{2m}\frac{\nu^2\pi^2}{3}\int dx :\rho^3(t,x):$, which leads to three-body δ-function interactions $\propto \delta(x_i - x_j)\delta(x_i - x_k)$ that may be ignored when the singular coincident-point behavior of wave functions is specified.]

(C) We now return to the model of Ref. [1] and examine it as a classical field theory. The equation of motion that follows from (5)–(7) reads

$$i\hbar\partial_t\Psi = -\frac{\hbar^2}{2m}(\partial_x \pm i\kappa^2\rho)^2\Psi \pm \hbar\kappa^2 j\Psi\,, \tag{17}$$

where the current j is given by

$$j = \frac{\hbar}{2im}[\Psi^*(\partial_x \pm i\kappa^2\rho)\Psi - \Psi(\partial_\kappa \mp i\kappa^2\rho)\Psi^*] \tag{18}$$

and satisfies a continuity equation with ρ:

$$\dot\rho + j' = 0\,. \tag{19}$$

Next we redefine the Ψ field by

$$\Psi(t,x) = e^{\mp i\kappa^2\int^x dy\,\rho(t,y)}\psi(t,x) \tag{20}$$

(the lower limit is immaterial, as it affects only the phase of ψ) so that

$$i\hbar\partial_t\psi(t,x) \pm \hbar\kappa^2\int^x dy\,\dot\rho(t,y)\psi(t,x) = -\frac{\hbar^2}{2m}\psi''(t,x) \pm \hbar\kappa^2 j(t,x)\psi(t,x)\,. \tag{21}$$

But the integral on the left side may be evaluated with the help of (19), and taken to the right, leading to our final equation, which is a Schrödinger equation with a *current* (j) nonlinearity:

$$\begin{aligned} i\hbar\partial_t\psi &= -\frac{\hbar^2}{2m}\psi'' \pm 2\hbar\kappa^2 j\psi\,,\\ j &= \frac{\hbar}{2im}(\psi^*\psi' - \psi\psi^{*\prime})\,. \end{aligned} \tag{22}$$

This is to be contrasted with the familiar nonlinear Schrödinger equation, where the nonlinearity involves the *charge density* (ρ):

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\psi'' - \lambda\rho\psi\,. \tag{23}$$

Equation (22) is similar to "derivative nonlinear Schrödinger equations" [with nonlinearity $\partial_x(\rho\psi)$ or $\rho\partial_x\psi$]. But unlike these, our equation apparently is not completely integrable [9]. [If instead of omitting $V(\rho)$ from (2), as we have done here, we choose $V(\rho) = -\frac{1}{2}\frac{\hbar^2\kappa^4}{m}\rho^3$, Eq. (17) acquires the extra term $-\frac{3}{2}\frac{\hbar^2\kappa^4}{m}\rho^2\Psi$ and becomes an integrable nonlinear Schrödinger equation with nonlinearity $\mp 2i\frac{\hbar^2\kappa^4}{m}\rho\partial_x\Psi$ [10]. However, the solitons are no linger chiral.]

Influenced by the known solutions to (23), we seek a one-soliton solution to (22) of the form

$$\psi = e^{-i(\omega t - kx)}\sqrt{\rho}\,. \tag{24}$$

With (24)

$$j = v\rho\,, \qquad v \equiv \frac{\hbar k}{m}\,, \tag{25}$$

so that our Eq. (22) coincides with (23) when λ is set equal to $\mp 2\hbar\kappa^2 v$. For positive λ, (23) possesses a single-soliton solution. In our case $\mp 2\hbar\kappa^2$ always has a definite sign, depending on the initial choice of sign in (3). Then (22) also possesses a single-soliton solution, provided v is chosen so that $\mp 2\hbar\kappa^2 v$ is positive. It is seen that the soliton of the Schrödinger equation (22), with a current nonlinearity, always moves in one direction (determined by the sign of v)—it is a chiral soliton. This is in contrast to the usual Schrödinger equation (23), with a charge-density nonlinearity, whose solitons can move on a line in both directions.

Henceforth, for definiteness, we take the lower sign and $\kappa > 0$; then the soliton solution exists for positive v, and takes the profile

$$\psi_{\text{soliton}} = \pm e^{i(mv/\hbar)(x-ut)}\frac{1}{\kappa}\sqrt{\frac{\hbar}{2mv}}\,\frac{\alpha}{\cosh\alpha(x - vt)}\,. \tag{26}$$

Here $u \equiv w/k$ and $\alpha^2 \equiv (m^2v^2/\hbar^2)(1 - 2u/v) > 0$, which is required to be positive, i.e., $u < v/2$.

The soliton's dynamical parameters may be evaluated as follows. Setting

$$N = \int dx\,\rho\,, \tag{27a}$$

we find

$$N_{\text{soliton}} = \frac{\hbar\alpha}{\kappa^2 2mv} = \frac{1}{\kappa^2}(1 - 2u/v)^{1/2}. \tag{27b}$$

The field energy, in terms of the rephased field ψ is

$$E = \frac{\hbar^2}{2m}\int dx\,\psi^{*\prime}\psi' \tag{28a}$$

and with the solution (26) one has

$$E_{\text{soliton}} = \tfrac{1}{2}Mv^2, \tag{28b}$$

where

$$M = mN(1 + \tfrac{1}{3}\kappa^4 N^2)\,. \tag{29}$$

The field momentum has an unconventional form, owing to the lack of Galileo invariance

$$P = \int dx\,(mj + \hbar\kappa^2\rho^2) \tag{30a}$$

giving with (26)

$$P_{\text{soliton}} = Mv\,. \tag{30b}$$

We see from these expressions, which imply

$$E_{\text{soliton}} = \frac{v}{2} P_{\text{soliton}}\,, \tag{31}$$

that the soliton's characteristics are those of a nonrelativistic particle of mass M, moving with velocity v and composed of N "constituents." While the phase velocity u is arbitrary, the group velocity v must be positive and exceed $2u$.

When v is negative there exists a "kink" solution

$$\psi_{\text{kink}} = \pm e^{i(mv/\hbar)(x-ut)} \frac{1}{\kappa} \sqrt{\frac{\hbar}{2m|v|}}\, \beta \tanh \beta(x - vt)\,, \tag{32}$$

where now $\beta^2 = -(m^2 v^2/2\hbar^2)(1 + 2u/|v|) > 0$, which must be positive, i.e., $u < v/2$. ψ_{kink} interpolates between the two "vacua"

$$\psi_{\text{vacuum}} = \pm e^{i(mv/\hbar)(x-ut)} \frac{\beta}{\kappa} \sqrt{\frac{\hbar}{2m|v|}}\,. \tag{33}$$

Because ψ_{kink} does not fall off at large distances, the kink's dynamical characteristics, corresponding to (27a)–(30b), diverge. But if the kink's energy and momentum are defined by subtracting the corresponding vacuum values, one may still establish the relation

$$E_{\text{kink}} = \frac{v}{2} P_{\text{kink}}\,. \tag{34}$$

Semiclassical quantization of our solutions remains an open problem. We note that the solutions are neither static nor periodic; hence new quantization techniques need to be developed. Here we observe the following. If in the quantum theory one replaces the Hamiltonian (6) by its normal ordered version, there are no infinities in the consequent N-body quantum mechanical problem. The two-body equation, for relative motion, reads instead of (10)

$$\left[-\frac{\hbar^2}{m} \partial_x^2 + \frac{P^2}{4m} - \frac{P}{m} \hbar \kappa^2 \delta(x) \right] u(x) = E u(x)\,. \tag{35}$$

Unlike (10), this possesses a bound state, with

$$E = \frac{P^2}{4m} (1 - \kappa^4)\,, \tag{36}$$

provided P/m is positive. If P/m is taken proportional to v, this condition is the same as that for the existence of the soliton. So we suspect that there is a relation between the classical soliton and quantum bound states.

Justification for this conjecture may be seen from the following argument. If in (36) we substitute (28b) and (30b) we get

$$M = \frac{2m}{1 - \kappa^4}\,. \tag{37}$$

On the other hand, if Eq. (29) is to be modified in the same way that one-loop quantum effects modify formulas for the cubic Schrödinger equation [11], we should replace (29) by

$$M_{\text{semiclassical}} = mN + \tfrac{1}{3} m \kappa^4 (N^3 - N)\,. \tag{38a}$$

For $N = 2$ this gives

$$M_{\text{semiclassical}} = 2m(1 + \kappa^4)\,, \tag{38b}$$

which agrees with (37) at "weak coupling," i.e., small κ. Although explicit solutions of the N-body quantum Schrödinger equation are not known for $N > 2$, one may establish perturbatively in κ that (38a) is consistent with the quantum bound states. Details will be presented elsewhere.

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