Steady Viscous Flow with Fractal Power Spectrum

Michael A. Zaks, Arkady S. Pikovsky, and Jürgen Kurths
Max-Planck Arbeitsgruppe für Nichtlineare Dynamik an der Universität Potsdam,
Am Neuen Palais 19, PF 601553, D-14415, Potsdam, Germany
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We demonstrate a family of two-dimensional steady viscous flows which have singular continuous (fractal) Fourier spectra. Such flows represent a novel intermediate stage between order and Lagrangian chaos: The motion of individual fluid particles in them is neither entirely correlated nor completely disordered. In the considered setup these flows are presented by the exact solutions of the Navier-Stokes equations and occupy a parameter subset of positive measure. Onset of this unusual state follows the formation of steady eddies and is caused by the development of singularities of return times along the particle paths near the stagnation points. [S0031-9007(96)01707-3]

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To get a feeling of historic affinity between hydromechanics and the theory of dynamical systems, one has only to recall several words like “sink,” “source,” and last but not least, one of the most important ones, the “flow.” When one views a steady motion of incompressible fluid as a dynamical system, the flow is really a phase flow, streamlines become phase trajectories, and the time independent spatial pattern recovers the geometry of the attractor. For this purpose, the Lagrangian description of the fluid motion seems more suited than the Eulerian one. Whereas the latter is bound to the fixed place in the space (where by virtue of steadiness the values of all observables remain time independent), the former traces the individual “fluid particles” along their paths and can deliver the picture of chaotic motion with its exponential growth of the distance between the initially close particles [1]. The stirring and mixing imposed by chaotic streamline patterns is of importance for geophysics and magnetohydrodynamics; for the numerous examples of Lagrangian chaos in various steady and time-dependent flow patterns, see, e.g., the monograph [2].

For obvious reasons the phase space of a chaotic autonomous dynamical system is at least three dimensional. Consequently, Lagrangian chaos can be encountered only in fully three-dimensional steady fluid motions. At first sight it may seem that the highest temporal complexity attainable for tracers transported by two-dimensional steady flows is periodicity or at most (in appropriate geometry) quasiperiodicity, with well pronounced correlations and discrete temporal spectra. However, slowing down near the stagnation points generates singularities in the turnover times of individual fluid particles. As we will demonstrate, this can result in the onset of some intermediate phase between order and chaos, where the power spectrum is (singular) continuous, but certain correlations persist over arbitrarily long times (recall that the absolutely continuous spectrum is a signature of chaos, but a dynamical system is mixing if and only if the correlations asymptotically decay [3]). This peculiar type of dynamics is known basically from the mathematical models of systems with incommensurate scales or quantum systems in quasiperiodic potentials (see, e.g., [4,5]), where, unlike our case, the singular spectra describe the spatial structure; to our knowledge it has never been previously reported in the context of fluid mechanics.

In this Letter we report on the class of two-dimensional steady viscous flows generated by time-independent forcing. Increase of the forcing amplitude leads to changes in the topology of streamlines pattern; from the Lagrangian point of view this marks the transition from the flow with a discrete temporal spectrum to the motion with a fractal one. With the help of the autocorrelation function, we obtain numerical estimates for the correlation dimension of the spectral measure.

Let the incompressible fluid with density ρ and viscosity ν flow over the 2-torus ($0 \le x \le 2\pi$, $0 \le y \le 2\pi$) under the action of the time-independent force $\boldsymbol{F} = (f \sin y, f \sin x, 0)$. (For the experimental realization of spatially periodic forcing in two-dimensional flows, see, e.g., [6–8].) The Navier-Stokes equations governing the fluid motion are

$$\frac{\partial}{\partial t}\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{\nabla P}{\rho} + \nu \nabla^2 \boldsymbol{v} + \boldsymbol{F}, \qquad (1)$$
$$\nabla \cdot \boldsymbol{v} = 0,$$

where $\boldsymbol{v}$ and P are, respectively, the velocity and the pressure. The structure of the forcing term is reminiscent of the Kolmogorov flow [9]. We restrict ourselves to the two-dimensional formulation; besides, the geometry of the domain implies that not only the forcing, but also the velocity field itself is periodic:

$$\boldsymbol{v}(x,y) = \boldsymbol{v}(x + 2\pi, y) = \boldsymbol{v}(x, y + 2\pi). \qquad (2)$$

Thus the possible perturbations are confined to the torus size, and the long-wave disturbances which are known to be the first to destabilize the Kolmogorov flow [9,10] are precluded. As a further difference from the Kolmogorov flow we prescribe the fixed nonzero mean flow across the

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domain in both directions, parametrizing it by the two flow rates α and β, respectively:

$$\int_0^{2\pi} v_x\, dy \,\Big|_{x=0,2\pi} = 2\pi\alpha,$$
$$\int_0^{2\pi} v_y\, dx \,\Big|_{y=0,2\pi} = 2\pi\beta\,. \tag{3}$$

Incompressibility of the fluid allows one to project out the pressure by introducing the streamfunction $\Psi(x,y)$: $v_x = \partial\Psi/\partial y$, $v_y = -\partial\Psi/\partial x$. The pattern, described by the steady solution

$$\Psi = \alpha y - \beta x + \frac{f\,\sin(x-\phi_1)}{\sqrt{\alpha^2+\nu^2}} - \frac{f\,\sin(y-\phi_2)}{\sqrt{\beta^2+\nu^2}}, \tag{4}$$

$$\phi_1 = \arctan\frac{\nu}{\alpha}, \qquad \phi_2 = \arctan\frac{\nu}{\beta}$$

of Eqs. (1) satisfying (2) and (3), yields the stationary velocity field

$$v_x = \alpha - \frac{f\,\cos(y-\phi_2)}{\sqrt{\beta^2+\nu^2}},$$
$$v_y = \beta - \frac{f\,\cos(x-\phi_1)}{\sqrt{\alpha^2+\nu^2}}.$$

In the absence of forcing ($f = 0$) the streamlines are straight and the velocity is everywhere the same; one has the trivial flow on the 2-torus with the rotation number α/β. [Of course the rotation number equals α/β also in the general case $\Psi = \alpha y - \beta x + \Phi(x,y)$, where $\Phi(x,y)$ is periodic in both arguments and bounded.]

The increase of the forcing amplitude f distorts the streamlines [Fig. 1(a)]. However, in a range of values of f the Lagrangian dynamics does not alter qualitatively: If α/β is rational, the streamlines are eventually closed; otherwise each particle path is dense in the domain. In both cases the Fourier spectrum of an observable $\xi(t)$ measured along the trajectory of the particle is discrete; the autocorrelation function $C(\tau) = \langle\xi(t)\xi(t+\tau)\rangle/\langle\xi^2(t)\rangle$ displays peaks which approach 1 for the time values corresponding to the multiples of period in the former case and to the denominators of rational approximations to α/β in the latter. This can be seen in Fig. 2(a); here and below we fix the values $\nu = 1$ and the "golden mean" rotation number $(\sqrt{5}-1)/2$.

At reaching the threshold value of the forcing amplitude

$$f = f_{\mathrm{cr}} = \sqrt{\alpha^2\beta^2 + \nu^2\max(\alpha^2,\beta^2)}, \tag{5}$$

two cusps appear at two particular streamlines, with the velocity vanishing at the cusp tips [Fig. 1(b)]. For $f > f_{\mathrm{cr}}$ each cusp tip splits into a couple of stagnation points: the elliptic one and the hyperbolic one (the Poincaré index [11,12] being 1 for the former and -1 for the latter). The pattern of streamlines (4) acquires new features: Along with the "global" particle paths crossing the whole domain, there appears a "localized" component which is built of two mutually symmetric isolated eddies [Fig. 1(c)]. Each eddy has the elliptic point at its center and is encircled by one of the separatrices of the respective hyperbolic stagnation point. Inside the eddies the particle paths are obviously closed; respectively, the above statement concerning the density of orbits under irrational values of α/β holds everywhere *outside* the eddies [13].

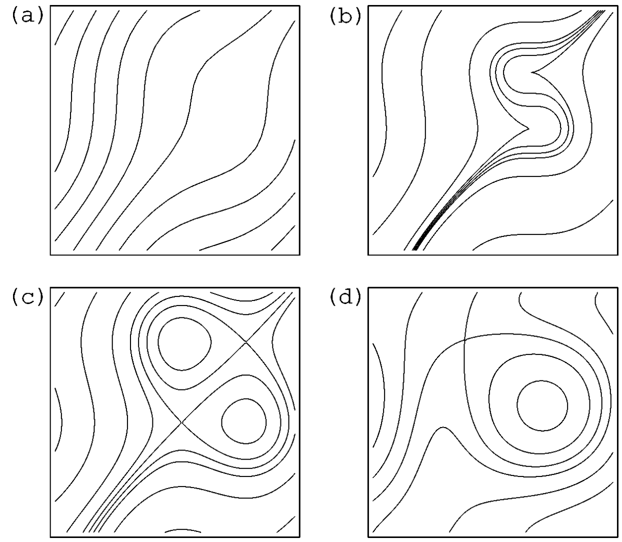

FIG. 1. Flow patterns for $\alpha = (\sqrt{5}-1)/2$, $\beta = 1$: (a) pattern (4) for $\nu = 1$, $f = \frac{1}{2}f_{\mathrm{cr}} = 0.587785$; (b) pattern (4) for $\nu = 1$, $f = f_{\mathrm{cr}}$; (c) pattern (4) for $\nu = 1$, $f = \frac{3}{2}f_{\mathrm{cr}} = 1.7633557$; (d) pattern (6).

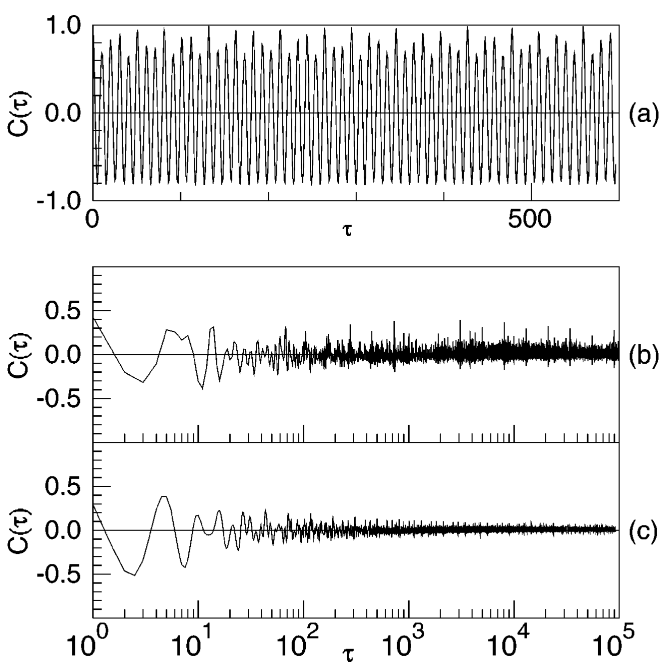

FIG. 2. Autocorrelation function for "fluid particles" in steady flow patterns: (a) pattern (4), parameter values as in Fig. 1(a); (b) pattern (4), parameter values as in Fig. 1(c); (c) pattern (6).

Let us choose a secant which is transversal to the global component of the flow. Because of the Hamiltonian nature of the flow (which is ensured by the incompressibility of the fluid), the Poincaré return mapping which the particle paths induce on this secant is conjugate to the shift on the circle: $\phi_{i+1} = (\phi_i + 2\pi\,\alpha/\beta) \bmod 2\pi$, both for rational and irrational values of α/β.

Under irrational values of α/β the power spectrum for any orbit of this map is discrete, and the motion is completely correlated, with the appropriate peaks of the autocorrelation function (ACF) tending to 1, like those in Fig. 2(a). However, the correlation properties of the flow appear to be remarkably different from those of its own Poincaré mapping. As can be seen from Fig. 2(b), unlike the subcritical case from Fig. 2(a), the highest peaks of the ACF do not even reach $\frac{1}{2}$; i.e., the dynamics is far from being completely ordered. The reason for the weakening of correlations lies in the nonuniform slowing down of motion in vicinities of the stagnation points. Take a smooth curve l transversal to the local stable separatrix W_s of a stagnation point. The turnover time τ as a function of a coordinate ζ on l diverges at the point ζ_0 where l and W_s intersect: $\tau(\zeta) \sim -\log|\zeta - \zeta_0|$. Consider two initially close fluid particles near W_s which move along two streamlines on the same side of W_s and slow down while approaching the stagnation point. The slowing is more pronounced for the particle which lies closer to the separatrix; it stays in the vicinity of the stagnation point longer than its counterpart, and the distance between them grows. The much stronger effect is observed for the two particles lying on the *opposite* sides of W_s, since one of them is doomed to hover in the stagnation region twice (for the first time on entering it along W_s, and for the second after making a tour around the eddy) and thereby gets a very noticeable lag. As a consequence the coefficients before the logarithmic terms in $\tau(\zeta)$ to the left and to the right from ζ_0 differ by a factor of 2 [13].

Albeit relatively low, the correlations do not entirely decay: At certain time values (which apparently mark the denominators of the best rational approximations to α/β) the ACF displays distinct peaks whose height remains above the certain finite level. Although with growth of time the intervals between these peaks are getting larger and larger (note the logarithmic scale along the time axis), they are well discernible at the largest computationally attainable times which correspond to tens of thousands of revolutions around the torus. The "vague" memory of this kind is characteristic for dynamical systems with singular continuous (fractal) power spectra [14]. The latter deliver a link between systems with the discrete (pure point) spectra and those with absolutely continuous spectra: Although the spectrum is continuous, the spectral measure is located on a set of zero Lebesgue measure. For the typical frequency values ω the Fourier sums $S(\omega, L) = L^{-1} \sum_{k=1}^{L} |\xi_k e^{i2\pi k\omega}|^2$ neither grow linearly with the increase of the sample length L, as would be the case for the δ peaks of the discrete spectrum, nor converge to constant values, as in the case of the absolutely continuous spectrum. In this sense, the conventional procedure of computing the power spectrum $S(\omega)$ from time series of progressively growing length provides the more and more "fractalized" approximations to the ultimate singular continuous object. Since the ACF is the Fourier transform of the power spectrum, the fractal properties of the latter can be evaluated with the help of the former. Thus the integrated squared autocorrelation function $C_{\text{int}}(t) = \frac{1}{t}\int_0^t |C(\tau)|^2\, d\tau$ for a state with purely singular continuous spectrum should decay as $\sim t^{-D_2}$ where D_2 is the correlation dimension of the spectral measure [15]. The plot of $C_{\text{int}}(t)$ corresponding to Fig. 2(b) is presented in Fig. 3; it provides evidence that the contribution of the discrete spectral component, if present at all, is extremely weak; the slope of this curve yields the estimate for D_2 of the fractal spectral measure: $D_2 = 0.51 \pm 0.02$.

The spectral properties of the Hamiltonian flow on a torus without fixed points ($f < f_{\text{cr}}$ in our terms) were first addressed by Kolmogorov [16], who showed that the spectrum was discrete for irrational rotation numbers, "not too fast" approximated by rationals, and conjectured that it could be continuous otherwise. Further, the possibility of weak mixing for the latter rotation numbers (which constitute a subset of zero measure on the set of all numbers) under certain conditions imposed on the distribution of bounded turnover times was demonstrated by Shklover [17]. In the presence of fixed points the return times are unbounded; as shown by Kochergin [18] in this case the flow does not mix if the prefactors before the logarithmic terms in the return times are balanced. This is precisely the case for the pattern (4) with two symmetric eddies where the sums of prefactors corresponding to the passage to the left and to the right from the stagnation points are obviously equal. Noteworthy, mixing must be present [19] in the threshold case $f = f_{\text{cr}}$ when the stagnation points are degenerate and the return times diverge in a powerlike way $\tau(\zeta) \sim |\zeta - \zeta_0|^{-1/6}$.

Applying the more elaborate periodic forcing, it is possible to excite a velocity field with a single eddy; a typical flow pattern with a streamfunction

$$\Psi(x, y) = \alpha y - \beta x + \sin x \cos y - \sin x - \cos y \quad (6)$$

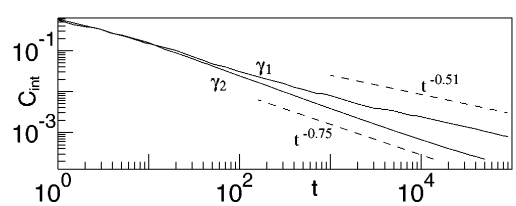

FIG. 3. Integrated autocorrelation for states from Fig. 1(c) (curve γ_1) and Fig. 1(d) (curve γ_2); cf. power laws shown by dashed lines.

is presented in Fig. 1(d). For this pattern the logarithmic singularities in the return times are not mutually compensated. The respective ACF is plotted in Fig. 2(c); although one notices many similarities to Fig. 2(b), the slow but definite decrease of the highest peaks is not to be overlooked. The decay of correlations implies mixing; presence of mixing for Hamiltonian flows on a torus with nonbalanced logarithmic singularities was shown by Sinai and Khanin [20]. As compared to the case of symmetric eddies, $C_{\mathrm{int}}(t)$ decays faster (Fig. 3); evaluation of the fractal dimension of the spectral measure results in $D_2 = 0.75 \pm 0.01$.

As regards the transport of the nondiffusive scalar tracers in these velocity fields, one may expect something "in between" the ordered motion over the regular pattern and the chaotic advection imposed by Lagrangian turbulence. Because of incompressibility, both Lyapunov characteristic exponents vanish identically; thereby the exponential divergence of close particles is prohibited, and the slower effects should be looked for. To enable the unbounded drift, we considered the periodic tiling on the plane and computed the value of

$$d_2(t) = \langle [x(T + t) - x(t) - t\overline{v}_x]^2 \rangle \tag{7}$$

averaged with respect to the time T along the particle paths, for both the symmetric field (4) and asymmetric pattern (6); here $\overline{v}_x$ is the mean velocity of the drift in the x direction. In both cases after a short intermediate stage with the time scale of one revolution around the torus, one observes the slow stretching: $d_2(t) \sim t^{2\chi}$ with $\chi \approx 0.07$ for the former and $\chi \approx 0.09$ for the latter.

Summarizing, the discussed class of steady flows is a reasonable candidate for the role of a state with the most complicated Lagrangian dynamics which can be observed in a time-independent two-dimensional setup. Noteworthy, for these flow patterns the property of having a singular continuous component in the spectrum is generic, unlike the case of dissipative dynamical systems where until now such spectra were reported only for certain marginal situations on the border between order and chaos [14]. Properly rewritten, the equations are governed by three dimensionless parameters: The forcing strength and two "Reynolds numbers" characterizing the mean flow in x and y directions, respectively. Obviously, each particular state with an irrational rotation number is structurally unstable, but the set of all these states taken together occupies almost the whole parameter space (its complement, corresponding to rational rotation numbers has zero measure). In other words, a randomly chosen point in the parameter space corresponds with probability 1 to a pattern with irrational rotation number. Thus one is tempted to conjecture that in the presence of the imposed mean flow a motion of a passive tracer along the typical two-dimensional steady pattern with isolated eddies possesses the fractal Fourier spectrum.

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