Localization in Two Dimensions, Gaussian Field Theories, and Multifractality

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We calculate nonperturbatively the multifractal scaling exponents of the critical wave function for two dimensional Dirac fermions in the presence of a random magnetic field. We do so by arguing that the multifractal scaling exponents can be expressed in terms of the free energy of random directed polymers on a Cayley tree. We find a weak-strong disorder transition for the multifractal scaling exponents of the wave function that is parallel to the freezing or glassy transition of the random polymer model. [S0031-9007(96)01600-6]

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Starting with the work of Wegner, it has been proposed that at the localization transition the critical wave functions are multifractal [1]. Early predictions of multifractality in the theory of localization have relied on renormalization group arguments applied to various nonlinear σ models. Although the validity of such perturbative approaches has been questioned [2], a consensus has developed on the multifractal nature of wave functions at the localization transition on the basis of numerical simulations [3].

In this Letter, we will approach multifractality at the localization transition from the following perspective. We consider Dirac fermions moving in a plane and with a static random magnetic field normal to the plane. The pure model can be derived by taking the continuum limit of various tight-binding Hamiltonians. Examples are the Chalker-Coddington network model, d-wave superconductors, and degenerate semiconductors [4], for which, typically, the density of states for the pure system has a V-shaped singularity at the Fermi energy. General considerations on the conductivity tensor [5] and calculations of the inverse participation ratio [6] predict that the random magnetic field localizes wave functions with energies close to the Fermi energy, whereas *exactly* at the Fermi energy the (critical) wave functions remain extended. Thus the model describes a metal-insulator transition in two dimensions.

A very useful property of this model is that the wave function at the Fermi energy can be calculated exactly for any realization of the random magnetic field $b(\mathbf{x}) = \nabla^2\Phi(\mathbf{x})$, and is given by $\psi(\mathbf{x}) = \exp[-\Phi(\mathbf{x})]$ [7]. It has been suggested [8] that the multifractal properties of the critical wave functions ψ are closely related to those of the so-called prelocalized states of a two-dimensional metallic cavity [9]. We will view the Dirac fermion probability density $\propto|\psi|^2$ as a random surface with a distribution controlled by the disorder. We will explore the connection between spatial averages of powers of the density and the partition function for random directed polymers. From this point of view, we will be able to explore nonperturbatively the multifractal nature of the critical wave functions. We will assume that the random magnetic field $b(\mathbf{x}) = \nabla^2\Phi(\mathbf{x})$ is Gaussian distributed through $P[\Phi] \propto \exp\{-\frac{1}{2g}\int d^2x[\nabla\Phi(\mathbf{x})]^2\}$. Here the *dimensionless* variance g measures the disorder strength. Thus the total flux through the plane is constant in our model, although it may fluctuate locally.

The multifractal nature of the wave function is manifest in the complex scaling of the inverse participation ratios: $\mathcal{P}(q, \frac{a}{L}) = \int \frac{d^2x}{a^2}|\Psi(\mathbf{x})|^{2q}$, where $\Psi(\mathbf{x})$ is obtained by normalizing ψ. Roughly, one expects scaling with system size $\mathcal{P}(q, \frac{a}{L}) \sim (L/a)^{-\tau(q)}$, with a and L being microscopic and macroscopic cutoff lengths, respectively. More precisely, the exponents $\tau(q)$ should be obtained from ensemble averaging according to

$$\tau(q) = \lim_{a/L\to 0} \frac{\langle \ln \mathcal{P}(q, a/L)\rangle}{\ln(a/L)}. \tag{1}$$

The exponents $\tau(q)$ probe spatial variations of the wave functions, and a nonlinear dependence on q is the signature of intricate multifractal scaling. One of the results of this work is to show that $\tau(q)$ is a self-averaging quantity; i.e., it is independent of the disorder realization in the thermodynamic limit. Hence the ensemble average is redundant in Eq. (1).

The calculation of $\tau(q)$ is a complicated task. A simplified version is easily done assuming "ergodicity," i.e., that the space integrals in $\mathcal{P}(q, a/L)$ (including the normalization of the wave function) can be substituted by ensemble averages:

$$\int \frac{d^2x}{a^2} e^{-2q\Phi(\mathbf{x})} \to \left(\frac{L}{a}\right)^2 \langle e^{-2q\Phi(\mathbf{x})}\rangle .$$

In this case it is possible to calculate the scaling exponent $\tau^*(q)$ defined by

$$\mathcal{P}^*(q, a/L) = \frac{(L/a)^2\langle e^{-2q\Phi(\mathbf{x})}\rangle}{[(L/a)^2\langle e^{-2\Phi(\mathbf{x})}\rangle]^q} \sim \left(\frac{a}{L}\right)^{\tau^*(q)}. \tag{2}$$

One then finds the parabola $\tau^*(q) = (2 - \frac{g}{\pi}q)(q - 1)$ for each critical point labeled by g [6]. This is a remarkably simple result which is again characteristic of a log-normal distribution. The same parabolic approximation $\tau^*(q)$ was obtained in Ref. [5] using a naive replica limit

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to calculate $\tau(q)$. Unfortunately, it is clear that $\tau(q)$ must differ from $\tau^*(q)$ for large q, since one verifies that $\tau(q)$ can never decrease with q. Therefore we can conclude that ergodicity cannot be true (at least for *all* q), and that the naive replica limit is ill-defined.

In this Letter we obtain the full function $\tau(q)$ for all q. Our results allow us to understand the limitations of the parabolic approximation and the breakdown of the assumptions leading to it. We find that the "ergodic" assumption holds only for $|q| \leq q_c = \sqrt{2\pi/g}$.

We will gain physical insights into this result by relying on a parallel with a model for random directed polymers, on the basis of which we also find that $\tau(q)$ as defined in Eq. (1) is a self-averaging quantity. To this end, we find it convenient to interpret

$$Z(q) = \int \frac{d^2x}{a^2} e^{-2q\Phi(\mathbf{x})}$$

as a random partition function with the moment q playing the role of inverse temperature. This will allow us to draw analogies between $Z(q)$ and the random partition function in various random energy models introduced by Derrida [10], since Φ can be thought of as a random energy. The problem of calculating the scaling exponents in Eqs. (1) and (2) thus amounts to the calculation of the thermodynamic averages

$$\tau(q) = \lim_{a/L\to 0} \frac{1}{\ln(a/L)} [\langle \ln Z(q)\rangle - q\langle \ln Z(1)\rangle], \quad (3)$$

$$\tau^*(q) = \lim_{a/L\to 0} \frac{1}{\ln(a/L)} [\ln\langle Z(q)\rangle - q \ln\langle Z(1)\rangle], \quad (4)$$

respectively. In this paper we will also address the question of when quenched and annealed averages over $Z(q)$ agree, i.e., whether or not

$$\upsilon(q) = \lim_{a/L\to 0} \frac{\langle \ln Z(q)\rangle}{\ln(L/a)} \quad (5)$$

and

$$\upsilon^*(q) = \lim_{a/L\to 0} \frac{\ln\langle Z(q)\rangle}{\ln(L/a)}$$

are equal. We will show that $\upsilon(q)$ is self-averaging, so that the average in the definition Eq. (5) is redundant in the thermodynamic limit. The equality $\upsilon(q) = \upsilon^*(q)$ is then a statement on ergodicity: It means that the ratio between spatial and ensemble averages does not scale with system size (although it need not be unity).

We will give below strong evidences that

$$\upsilon(q) = \upsilon^*(q) = 2\left(1 + \frac{q^2}{q_c^2}\right), \qquad |q| \leq q_c = \sqrt{\frac{2\pi}{g}}. \quad (6)$$

In other words, for sufficiently small moments (high temperatures) the quenched and annealed averages of $Z(q)$ agree. For moments $|q| > q_c$, our analysis suggests a linear behavior

$$\upsilon(q) = 4\frac{|q|}{q_c}, \qquad |q| > q_c = \sqrt{\frac{2\pi}{g}}. \quad (7)$$

Our evidences for Eqs. (6) and (7) come from a mapping to a random polymer model and numerical calculations. Let us discuss them in turn.

$\upsilon(q)$ and $\upsilon^*(q)$ are more likely to be equal if fluctuations of $Z(q)$ are small. To study the strength of the fluctuations, we consider the following ratios of powers of the partition function:

$$R_n(q) = \frac{\langle Z^n(q)\rangle}{\langle Z(q)\rangle^n} \geq 1.$$

In the calculation, we will need the two-point correlation function

$$\langle \Phi(\mathbf{x})\Phi(\mathbf{y})\rangle = -\frac{g}{2\pi} \ln \frac{|\mathbf{x} - \mathbf{y}|}{L}, \qquad a \leq |\mathbf{x} - \mathbf{y}| \leq L. \quad (8)$$

With our choice of boundary condition in Eq. (8), we find that

$$R_n(q) \sim \int \frac{d^2x_1}{L^2} \cdots \frac{d^2x_n}{L^2} \prod_{i<j} \left| \frac{\mathbf{x}_i - \mathbf{x}_j}{L} \right|^{-2gq^2/\pi} \sim c_n(gq^2) + \left(\frac{a}{L}\right)^{2(n-1)(1-ngq^2/2\pi)}. \quad (9)$$

In the thermodynamic limit, the right hand side is a finite number for $n \leq (2\pi)/(gq^2)$ (assuming $n > 1$). In this case $Z^n(q)$ fluctuates weakly. But for $n > 2\pi/gq^2$, $R_n(q)$ diverges, and thus $Z^n(q)$ fluctuates strongly. There are important consequences that follow from Eq. (9): (i) There exists a *sequence of critical* q_n given by $q_n^2 = \frac{2\pi}{ng} \equiv \frac{q_c^2}{n}$, below which $\ln\langle Z^n(q)\rangle / \ln(a/L) = \ln\langle Z(q)\rangle^n / \ln(a/L)$. The appearance of these critical moments is due to the competition of two terms in Eq. (9). The first term is entropic and comes from integrating over phase space. The second comes from the short distance enhancements of the two-point function in Eq. (8). Remarkably, the *exactly same* sequence of critical moments is shared by the random energy models studied in [10]. This example demonstrates the origin of the critical moments. (ii) Caution is needed when using the replica trick $\langle \ln Z(q)\rangle = \lim_{n\to 0} \frac{\langle Z^n(q)\rangle - 1}{n}$. Indeed $\langle Z^n(q)\rangle$ is not an analytic function of n due to singularities at $n = 2\pi/q^2 g$ and $n = 1$, and caution must be used when using the replica trick to calculate $\upsilon(q)$.

We will now draw a parallel between our problem of localization and a model of random directed polymers studied by Derrida and Spohn [11]. We will consider a K-nary tree. The polymer model is defined by first assigning random variables ϕ_{ij} to each branch of a tree made of N levels. The first index i labels the levels in the tree, the second one j labels any of the K^i branches

on a given level i. A random energy is then assigned to all K^N directed paths $\mathcal{P}$ on the tree according to $\Phi_N(\mathcal{P}) = \sum_{ij\in\mathcal{P}} \phi_{ij}$. An example of a binary tree with two levels is shown in Fig. 1.

The polymer model is related to the random wave function $e^{-\Phi(\mathbf{x})}$ in the following way. We first partition the macroscopic volume L^2 into K boxes of equal size. The same partitioning is then iterated N times, thereby generating K^N different boxes with the microscopic area $a^2 = K^{-N}L^2$. Each box can be associated with a directed path $\mathcal{P}$ on the K-nary tree and can thus be labeled by $\mathcal{P}$. In this way, the polymer energy $\Phi_N(\mathcal{P})$ can be viewed as a function of $\mathbf{x}$ and is related to $\Phi(\mathbf{x})$. In the following we will show that by properly choosing the distribution of ϕ_{ij}, the random function $\Phi_N(\mathcal{P})$ shares many known properties with the random function $\Phi(\mathbf{x})$.

To study the relation between the two random distributions $\Phi_N(\mathcal{P})$ and $\Phi(\mathbf{x})$, we consider the two-point function $\langle\Phi_N(\mathcal{P})\Phi_N(\mathcal{P}')\rangle$. It depends on the choice for the probability distribution of the random variables ϕ_{ij}. It will also depend on the distance between the two paths $\mathcal{P}$ and $\mathcal{P}'$ on the tree. Such a distance is called an ultrametric. It essentially counts the number of common branches $I_{\mathcal{P}\mathcal{P}'}$ shared by the two paths and is defined by $d(\mathcal{P},\mathcal{P}') = K^{-\frac{I_{\mathcal{P}\mathcal{P}'}}{2}}$. Anticipating a little, $d(\mathcal{P},\mathcal{P}')$ is related to the Euclidean distance $|\mathbf{x}-\mathbf{x}'|$ through

$$\ln d(\mathcal{P},\mathcal{P}') \approx \ln\frac{|\mathbf{x}-\mathbf{x}'|}{L} + \mathcal{O}(1). \qquad (10)$$

To make a connection with our problem of localization, we require that the theory on the tree be Gaussian; i.e., we are considering only Gaussian probability distributions of the ϕ_{ij} although possibly with varying widths. Such models have been systematically studied in [10–12]. We then require that $\langle\Phi_N(\mathcal{P})\Phi_N(\mathcal{P}')\rangle$ depends *logarithmically* on the ultrametric, very much in the same way that the two-point function in Eq. (8) depends logarithmically on the Euclidean metric. The latter condition is very restrictive. It can only be satisfied if all ϕ_{ij} are identically distributed Gaussian random variables of width g_t, in which case one easily verifies that

$$\langle\Phi_N(\mathcal{P})\Phi_N(\mathcal{P}')\rangle = -\frac{2g_t}{\ln K}\ln d(\mathcal{P},\mathcal{P}'). \qquad (11)$$

Equation (10) then follows with the help of Eq. (8), and by requiring that $g_t = [\ln K/(4\pi)]g$. Note that a Gaussian field theory on a tree with two-point functions depending algebraically on the ultrametric is easily implemented when the width depends on the level i of the tree.

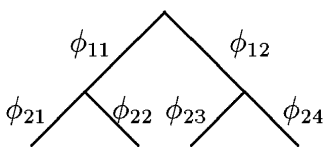

FIG. 1. K-nary tree with $K = 2$ and two levels.

All n-point correlations in the tree model depend exactly in the same way on the ultrametric as their counterparts in the continuum model do on the Euclidean metric, since both theories are Gaussian with logarithmic two-point functions. Hence the n-point functions in the random polymer model and in the continuum field theory have the *same* scaling exponents. We can also calculate $\langle Z_N(q)\rangle$ and $\nu_{\text{tree}}(q)$ in the random polymer model, where $Z_N(q) = \sum_{\mathcal{P}}\exp[-2q\Phi_N(\mathcal{P})]$, and $\nu_{\text{tree}}(q) = \lim_{N\to\infty}\frac{2}{\ln K}\frac{\ln Z_N(q)}{N}$. Derrida and Spohn [11] have studied $Z_N(q)$ and $\ln Z_N(q)$ under the condition Eq. (11) and calculated $\nu_{\text{tree}}(q)$ exactly. The moments $\langle Z_N^n(q)\rangle$ yield the same sequence of critical temperatures as obtained from the ratios $R_n(q)$ in Eq. (9) and Eqs. (6) and (7) are obtained, provided we choose $g_t = [\ln K/(4\pi)]g$. Moreover, it is known that $\ln Z_N(q)/N$ is a self-averaging function in the limit $N\to\infty$ [12]. We would like to stress that the above results are independent of K. This indicates that tree models with different K produce distributions $\Phi_N(\mathcal{P})$ with the same scaling properties. In other words, the scaling properties are not affected by different choices of ultrametrics. In turn, this strongly suggests that the scaling exponent $\nu(q)$ is not affected by the differences between the ultrametrics and the Euclidean metric. To support this hypothesis, we present below Monte Carlo (MC) calculations of the scaling exponent $\nu(q)$ for the $\mathcal{L} = \frac{1}{2g}(\nabla\phi)^2$ field theory.

We work on a square lattice made of 512×512 sites and choose periodic boundary conditions. In Fig. 2, we have plotted the four realizations of $\nu_{\text{MC}}(q) \equiv \ln Z_{\text{MC}}(q)/\ln(512)$ in a log-log plot. Also shown is the exact result for the tree $\nu_{\text{tree}}(q) = \ln Z_N(q)/\ln K^{N/2}$. Two facts are immediately obvious. First, within statistical fluctuations from finite size effects, the four MC realizations fall right on top of the exact result for the tree model. Second, there appears to be the precursor of a discontinuity in the second derivative of $\nu_{\text{MC}}(q)$ as indicated in Fig. 2 (inset). In this regard, there are noticeable statistical fluctuations in a narrow crossover region from quadratic to linear behavior, as one would expect from a phase transition. In summary, based on the parallel between the tree model and the $\mathcal{L} = \frac{1}{2g}(\nabla\phi)^2$ field theory, as well as on the MC evidence above, we *conjecture* that $\nu(q)$ is self-averaging and that its dependence on q is quadratic for $|q| \le q_c$ and linear for $|q| > q_c$ with a discontinuity at q_c in its second derivative, as in Eqs. (6) and (7).

Next we assume the validity of our conjecture $\nu(q) = \nu_{\text{tree}}(q)$, and apply it to the calculation of $\tau(q)$. There are *two* distinct regimes depending on the strength g of the disorder. In the weak disorder regime defined by $q_c > 1$, we find

$$\tau(q) = \begin{cases} 2(1-\frac{\text{sgn}\,q}{q_c})^2 q, & q_c < |q|, \\ 2(1-\frac{1}{q_c^2}q)(q-1), & q_c \ge |q|, \end{cases}$$

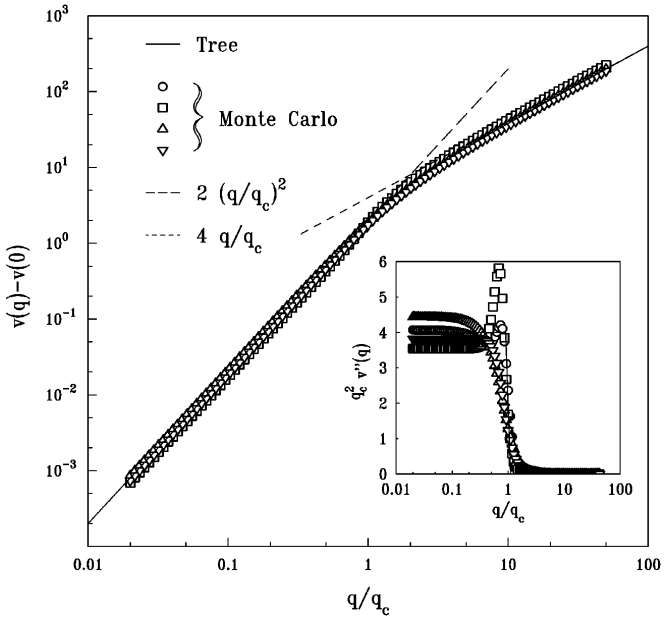

FIG. 2. $v(q) = \ln Z(q)/\ln(L/a)$, as a function of q/q_c [$q_c = (2\pi/g)^{1/2}$]. The exact results for the tree model are shown together with four MC results for a discrete version of the $\mathcal{L} = \frac{1}{2g}(\nabla\phi)^2$ theory on a 512×512 square lattice. The MC results follow closely (with small fluctuations) the ones for the $N \to \infty$ tree, suggesting that $v(q)$ is a self-averaging quantity as predicted by the tree model. The dashed lines show the asymptotic behavior of $v(q)$ for large and small q. The inset shows the second derivative of $v(q)$, showing strong evidence for a discontinuity at q_c.

in the limit $a/L \to 0$. In the weak disorder regime, the parabolic approximation $\tau^*(q) = D^*(q)(q - 1)$, where $D^*(q) = 2 - (2/q_c^2)q$, is exact for all moments q (including some integer ones) satisfying $|q| \leq q_c$. Here the parabolic approximation (PA) is obtained from the *annealed* disorder average Eq. (4) instead of the quenched average Eq. (3). In the strong disorder regime defined by $q_c \leq 1$, the quenched and annealed averages of $Z(q)$ are unequal for all moments $q_c \leq |q|$ including the integer ones, and the PA completely breaks down:

$$\tau(q) = \begin{cases} \frac{4}{q_c}(q - |q|), & q_c < |q|, \\ -2(1 - \frac{q}{q_c})^2, & q_c \geq |q|. \end{cases}$$

In the strong coupling regime, the inverse participation ratios for all positive integer moments do not scale with system size, i.e., $\tau(q) = 0$. Such a behavior is usually interpreted as characteristic of a localized wave function. In this regard, it would be highly interesting to calculate the dc conductivity along the line $g > 0$. In the random directed polymer model, the phase transition at q_c is interpreted as freezing at low temperatures, since the heat capacity vanishes abruptly for $q > q_c$. In the problem of localization, the glassy regime *and* the condition of normalization of the wave function conspire to yield inverse participation ratios which do not scale with the system size for large values of the disorder strength $g > g_c \equiv 2\pi$. The value of the disorder strength $g_c = 2\pi$ signals that the quenched and annealed averages over the wave function normalization factor $Z(1)$ are not equal anymore.

The corresponding multifractal scaling exponent $f(\alpha)$ obtained through a Legendre transformation of $\tau(q)$ and defined on $d_- \leq \alpha \leq d_+$ is given by

$$f(\alpha) = 8\frac{(d_+ - \alpha)(\alpha - d_-)}{(d_+ - d_-)^2},$$

where $d_\pm = 2(1 \pm \sqrt{\frac{g}{2\pi}})^2$ for $g < 2\pi$ and $d_+ = 8\sqrt{\frac{g}{2\pi}}$, $d_- = 0$ for $g > 2\pi$. Notice that $f(\alpha)$ is positive.

To conclude, we believe that the tree model is the proper regularization of the Euclidean Gaussian field theory $\mathcal{L} = \frac{1}{2g}(\nabla\phi)^2$ with regard to the scaling exponents for the inverse participation ratios. It is an important open problem to investigate further (i) the validity of our *conjecture* that the exponents $v(q)$ are the same for the Euclidean field theory and for the field theory constructed on the tree and (ii) whether other scaling exponents agree in both models.

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