Parametric Variation of Chaotic Eigenstates and Phase Space Localization

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We investigate the phase space localization properties of eigenstates of a quantum system possessing a chaotic classical limit. Parametric variation of the system suggests introducing a measure of correlations between state overlap intensities and level velocities to infer information about the extent of eigenstate localization. Random matrix theory predicts no correlations. Yet when applied to the chaotic stadium billiard, we find large correlations reflecting the significant eigenstate scarring due to the parametric action variations of the orbits homoclinic to the central trajectory underlying the wave packet. The analysis can be applied to data taken with quantum dots in the Coulomb-blockade regime and microwave cavities. [S0031-9007(96)01650-X]

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Nearly twenty years ago, Berry [1] and Voros [2] made a conjecture on the nature of eigenstates in the semiclassical limit of a quantum system possessing a simple chaotic classical analog. To within quantum fluctuations, "chaotic" eigenstates should respect the ergodic hypothesis in phase space, $\delta(E - H(\mathbf{p}, \mathbf{q}))$, as applied to wave functions. Heller later modified this picture with his prediction of eigenstate scarring along shorter classical periodic orbits [3]. This initiated continuing investigations and debate as to the phase space "localization" or scarring properties of chaotic eigenstates. In this context, localization should be understood as meaning deviations from the ergodic expectation and the inherent quantum fluctuations. A need arose to make connections between quantum mechanics and phase space. Thus Wigner transforms, Husimi distributions, and wave packets have been employed in various contexts. The issues and consequences are far from being settled and in this Letter (i) we introduce a measure involving parametric variations that probes such eigenstate localization, (ii) show random matrix models reflecting wave function ergodicity predict vanishing correlations, (iii) use dynamical arguments to explain surprisingly large correlations found in the stadium billiard, and (iv) suggest analyzing the conductance data from quantum dots in the Coulomb-blockade regime [4] and data from microwave cavities [5].

The relevance to localization of looking at parameter dependencies can be viewed in the following manner. If a system's eigenstates satisfy the ergodic expectation, then a smooth perturbation will democratically connect all states near a given energy surface. The evolution of any one eigenstate or energy level over a large enough parameter range will be statistically equivalent to their respective neighboring states or levels. On the other hand, localization creates the opportunity for undemocratic behaviors. One example would be an excessive proportion (in a statistical sense) of some states being essentially disconnected by the perturbation from other states. This would lead to an increase, relative to statistical expectations, of short range avoided crossings. However, looking to the level statistics is not likely to be the most sensitive test of nonergodic behavior in the eigenstates. One needs to design a more adapted measure to probe such "nongeneric" behavior. Ahead we introduce a correlation function between overlap intensities and level velocities, which immediately suggests itself in strength functions' parametric behavior.

Consider a quantum system governed by a parameter dependent Hamiltonian, $\hat{H}(\lambda)$. Suppose that the dynamics are chaotic for all values of the λ parameter range of interest and no symmetry is being weakly broken. Then the statistical properties are expected to be stationary with respect to λ and we do not have to be concerned with the transition to or from chaos. In his original introduction of scarring, Heller [3] made use of a strength function

$$S(E, \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, e^{iEt/\hbar} \langle \phi \mid \phi(t; \lambda) \rangle = \sum_i p_{\phi i}(\lambda) \delta(E - E_i(\lambda)), \tag{1}$$

where $p_{\phi i}(\lambda) = |\langle \phi \mid E_i(\lambda) \rangle|^2$, $E_i(\lambda)$ labels the eigenstates and eigenvalues, $S(E, \lambda)$ is the Fourier transform of the autocorrelation function of a wave packet initial state $|\phi\rangle$, and $\overline{S}(E, \lambda)$ denotes the smooth part resulting from the Fourier transform of just the extremely rapid initial decay due to the shortest time scale of the dynamics (zero-length trajectories). We will take $|\phi\rangle$ to be a wave packet with phase space image $\rho_\phi(\mathbf{p}, \mathbf{q})$, but other choices are possible. If momentum space localization is the main interest, the natural choice would be a momentum eigenstate.

The presence of very large overlap intensities in $S(E, \lambda)$ indicates localization near $\rho_\phi(\mathbf{p}, \mathbf{q})$. In Fig. 1, we show $S(E, \lambda)$ for a wave packet initially oriented along the horizontal axis of the stadium billiard, a highly chaotic system. We see that the levels associated with the peaks do not appear to move democratically. This suggests introducing an overlap intensity-level velocity correlation

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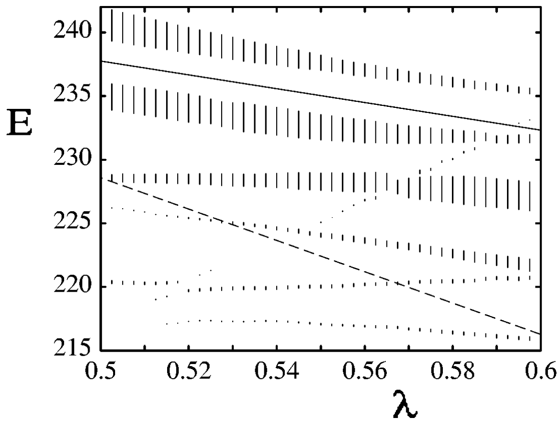

FIG. 1. Graphical representation of $S(E,\lambda)$ for the stadium. Each small line segment is centered on an eigenvalue and its lambda value. The heights are proportional to $p_{\phi i}(\lambda)$. λ is the ratio of the side length to semicircle diameter. As λ increases, the stadium stretches horizontally, but is scaled to preserve its area. $|\phi\rangle$ is a horizontally directed wave packet centered in the stadium and whose momentum is roughly set so that the largest dimension of the stadium spans 12 wavelengths. Over the range of λ plotted, the stadium lengthens approximately one wavelength at the mean momentum of $|\phi\rangle$. See the text for the solid and dashed curves.

coefficient, $C_\phi(\lambda)$, which is defined as

$$C_\phi(\lambda) = \left\langle \tilde{p}_{\phi i}(\lambda) \frac{\partial \tilde{E}_i(\lambda)}{\partial \lambda} \right\rangle_E . \quad (2)$$

It weights most the level velocities whose associated eigenstates possibly share common localization characteristics and measures the tendency of these levels to move in a common direction. In this expression, the tildes indicate that the variables are already zero-centered, and rescaled to unitless quantities with variance one. The set of states included in the local energy averaging can be left flexible except for the constraint that only energies where $\overline{S}(E,\lambda)$ is roughly constant can be used or some unfolding must be applied. $C_\phi(\lambda)$ thus has a simple form and the additional advantage of involving quantities of direct physical interest. Level velocities arise in thermodynamic properties and overlap intensities often arise in the manner used to couple into the system.

That $C_\phi(\lambda)$ measures localization is seen in two steps. First it can be calculated within random matrix theory which is supposed to apply generically to chaotic systems. It sets the reference point for one's expectations of the statistical properties and is a way of deriving the results for systems obeying the ergodic hypothesis. Let $\{\hat{H}(\lambda)\}$ be given by a Gaussian ensemble (GE) and be constructed as $\hat{H}(\lambda) = \hat{H}_1 + \lambda \hat{H}_2$, where $\hat{H}_1$ and $\hat{H}_2$ are independently chosen GE matrices. Because the GE is invariant under the set of transformations that diagonalize the ensemble, the choice of $|\phi\rangle$, although a fixed vector, is entirely arbitrary. The overlaps and level velocities are independent over the ensemble. With the overbar denoting ensemble averaging,

$$\overline{C_\phi(\lambda)} = \left\langle \overline{\tilde{p}_{\phi i} \frac{\partial \tilde{E}_i(\lambda)}{\partial \lambda}} \right\rangle_E = \langle \overline{\tilde{p}_{\phi i}} \rangle_E \left\langle \overline{\frac{\partial \tilde{E}_i(\lambda)}{\partial \lambda}} \right\rangle_E = 0 . \quad (3)$$

In fact, it is essential to keep in mind that *every* choice of $|\phi\rangle$ gives zero correlations within the random matrix framework. The existence of even a single $|\phi\rangle$ in a particular system that leads to nonzero correlations violates ergodicity. It is straightforward to go further and consider the mean square fluctuations of $C_\phi(\lambda)$,

$$\overline{C_\phi(\lambda)^2} = \left\langle \overline{\tilde{p}_{\phi i} \tilde{p}_{\phi j} \frac{\partial \tilde{E}_i(\lambda)}{\partial \lambda} \frac{\partial \tilde{E}_j(\lambda)}{\partial \lambda}} \right\rangle_E = \frac{1}{N} , \quad (4)$$

where N is the effective number of states used in the energy averaging. Again the level velocities are independent of the eigenvector components. The $i \neq j$ terms vanish due to the independence of the diagonal elements of the perturbation leaving only the diagonal terms that involve the quantities that respectively enter the variance of the eigenvector components and the mean square level velocity. The final result reflects the equivalence of ensemble and spectral averaging in the large-N limit. Therefore, in ergodically behaving systems, $C_\phi(\lambda) = 0 \pm N^{-1/2}$ for every choice of $|\phi\rangle$.

Next, we would like to have a picture of how localization can lead to nonzero correlations in $C_\phi(\lambda)$. It is simply reflecting finite time correlations in the classical dynamics. The classical propagation of $\rho_\phi(\mathbf{p},\mathbf{q})$ will relax to an ergodic long time average. However, during the relaxation, recurrences lead to localization in the eigenstates. See Heller's original arguments [3] for the existence of scarring. $p_{\phi i}$ weights most heavily the group of eigenstates localized near $\rho_\phi(\mathbf{p},\mathbf{q})$. Diagonal matrix elements (or level velocities) of some perturbation for these states may not fluctuate about the classical average of the perturbation over $\delta(E - H(\mathbf{p},\mathbf{q}))$. If not, $C_\phi(\lambda) \neq 0$. Note that this means some choices of $|\phi\rangle$ will still lead to zero correlations even though the system has localization. It takes only one nonzero result to demonstrate conclusively localization, but to obtain a complete picture, it is necessary to consider many $|\phi\rangle$ covering the full energy surface.

We apply the velocity-intensity correlation measure to the stadium billiard. In Fig. 1, all large overlaps moved down through the spectrum as λ increased. The correlations given in Fig. 2 show a nearly perfect average correlation coefficient of $C_\phi(\lambda) = -0.665$ (-1.0 is the lowest possible value). Furthermore, the value of N in this calculation is about 50. Considering Eq. (4), the result is quite inconsistent with ergodicity. We also find that there appears to be a significant structural persistence in the strength function. Many level crossings have no influence in redistributing the intensities.

At a heuristic level, the direction of the weighted level velocities is simple to understand. If a state has

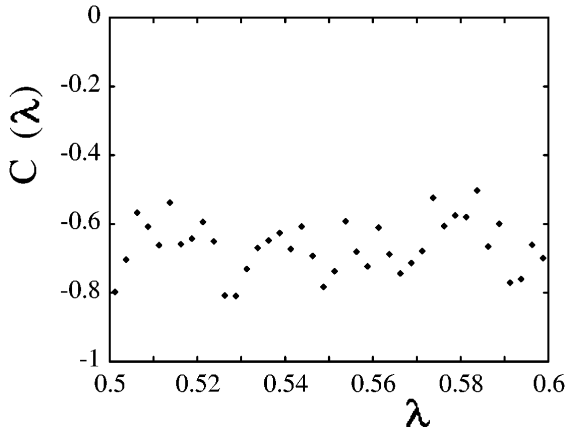

FIG. 2. $C_\phi(\lambda)$ versus λ. The average value is stationary at roughly -0.665 with the scatter consistent with $N^{-1/2}$ fluctuations. The same $|\phi\rangle$ is used as in Fig. 1.

enhanced intensity along the horizontal bounce periodic orbit and the stadium is smoothly lengthened, to maintain a constant number of wavelengths across implies reducing the momentum, i.e., decreasing the energy. An upper bound for the slope derives from a quantization condition along the periodic orbit and is shown as the dashed straight line in Fig. 1. The structural stability and large nonzero correlations can be explained more precisely in semiclassical theory. In a series of papers [6], it was shown that the strength function could be expressed as a sum over orbits homoclinic to the central orbit associated with $|\phi\rangle$ in the usual semiclassical form

$$S(E,\lambda) \approx \sum_j A_j \exp(iS_j/\hbar - i\nu\pi/4). \tag{5}$$

In this case, A_j cuts off the amplitude of homoclinic orbits crossing far from the centroid of $|\phi\rangle$. The key point is that the homoclinic orbits organize *all the return dynamics* in which trajectories initially nearby fall away from the central periodic orbit but later return to the neighborhood again. It was found that when all returning orbits were included whose periods do not exceed the Heisenberg time $[\tau_H = \hbar\bar{\rho}(E)]$, the summation converged well to the discrete quantum strength function.

In general, a perturbation will alter the value of the classical actions, S_j, and amplitudes, A_j, of each homoclinic orbit. We assume in the following that the mean level velocity due to the perturbation is zero. The summation is most sensitive to the changing actions because of the associated rapidly oscillating phases. To be consistent with zero correlations, the action variations must be "randomly" distributed about zero. Otherwise, shifting the energy surface partially compensates the perturbation effects and correlations are necessarily present. In fact, a simple condition can be derived in which the sum remains invariant to first order. Using τ_j (orbit period) $= \partial S_j/\partial E$, it turns out that if for every homoclinic orbit, $\tau_j \leq \tau_H$, a constant energy surface shift can be chosen such that this condition is closely satisfied,

$$\Delta E \equiv \Delta E_j = -\frac{\Delta\lambda}{\tau_j}\frac{\partial S_j(\lambda)}{\partial\lambda}, \tag{6}$$

then the strength function is just being rescaled in energy. All the overlaps will remain the same but move with the same constant velocity giving strong correlations and the aforementioned structural persistence of $S(E,\lambda)$. It is the dynamical correlations in the distribution of the set of $\{\tau_j^{-1}\partial S_j(\lambda)/\partial\lambda\}$ that is at the very origin of scarring in eigenstates. Without structural persistence, even if there existed a large intensity overlap somewhere, it would quickly disappear with a change in λ. Probabilistically then, the chances of finding large overlaps in a spectrum would be greatly reduced, or in other words, there would be far less localization.

For billiards, Eq. (6) simplifies to $2E$ times the logarithmic derivative with λ of the orbit lengths. In Fig. 3, we show the distribution of derivatives for the homoclinic orbits. The average shift can be used to calculate an average level velocity for the localized states, $\partial E/\partial\lambda$, from Eq. (6). The solid line in Fig. 1 shows the result and follows large amplitudes nicely as lambda varies. This gives a more complete explanation for the significant amount of scarring noted in the stadium billiard. Eventually at long times, the properties of the homoclinic orbits must equilibrate and for sufficiently small $\hbar$ (the long Heisenberg time limit), the ergodic expectations must be recovered. Significant localization should exist up to the point in the spectrum where the classical dynamical time scale during which the homoclinic manifolds uniformly explore the energy surface becomes much shorter than τ_H.

It is worthwhile to make some remarks on applying $C_\phi(\lambda)$ to the Coulomb-blockade conductance data [4];

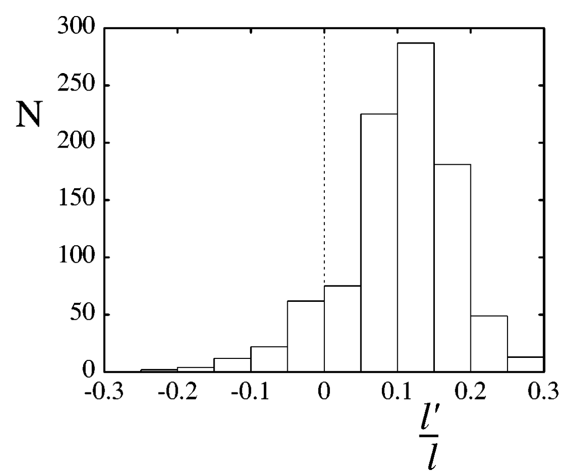

FIG. 3. Histogram of the logarithmic derivative of the first $\sim$1000 homoclinic orbits. The mean is 0.10. The horizontal bounce periodic orbit derivative is 0.27 and for the vertical bounce orbit -0.34. In Fig. 1, the barely visible, upwardly moving energy level has a velocity matching the vertical bounce orbit slope.

we have in mind the low temperature limit. There the peak conductances and resonance energies are tracked as a function of an external magnetic field or shape-changing gate voltages. The resonance energy variations are related to a single particle level velocity and the peak conductance is proportional to a quantity similar to $p_{\phi i}(\lambda)$ [7]. For example, consider a case where the electrons enter and exit the quantum dot through structures that are reflection symmetry related. There the expression for the conductance reduces to the form of Eq. (2). Even without symmetry, the random matrix theory will predict no correlations for the slightly more complicated form whereas localization of the eigenstates can generate nonzero correlations. If the tunneling contacts are placed at opportune sites such as at the opposing horizontal ends of a stadium billiard, some shape distortions will reveal the localization properties. Not every perturbation will. In the stadium example given here, changing the aspect ratio is perfectly adapted to testing for localization along the horizontal or vertical bouncing periodic orbits. However, this perturbation would not be expected to show correlations if $|\phi\rangle$ were placed on the self-retracing periodic orbit that has a V shape. We mention also that the microwave cavity data [5] can be studied with even more flexibility since they have measured the eigenstates and can therefore meticulously study a wide range of $|\phi\rangle$ to get a complete picture of the eigenstate localization properties.

$S(E, \lambda)$ contains all the information about the correlations and suggested the form of the measure we have introduced. The eigenstates can show a great degree of localization even in a spectral range where it is difficult to detect deviations from random matrix predictions for the spectral statistics. The stadium billiard shows large correlations in the intensity-velocity measure near the 300th–400th excited level but still has good agreement with random matrix spectral measures; this energy range is typical for experimentally produced quantum dots. In fact, spectral measures are more closely related to level curvatures. The intensity-curvature correlation for the stadium also gives the random matrix result zero.

The classical return dynamics organized by the homoclinic orbits is responsible for the excess localization occurring in the quantum system. This leads to the introduction of a new classical time scale over which the properties of the homoclinic orbits fully explore the energy surface. In a uniformly hyperbolic system, this time scale will reduce to known scales associated with the Lyapunov instability in the system. However, many of the paradigms used in chaotic systems are not uniformly hyperbolic and the exploration time scale will lengthen as some regions of phase space will be difficult to enter and trajectories will tend to get trapped there before returning nearby to their original starting points in phase space. There will be experimental, quantum mechanical consequences of these dynamical correlations which are fully described by the underlying homoclinic motion.

Finally, we mention that $C_\phi(\lambda)$ could be used to analyze integrable, near-integrable, or mixed phase space dynamics as well. For example in the mixed case, it might well provide new insight into "cantorus quantization" in the neighborhood of the frontier between a KAM region of regular motion and chaos [8]. In these cases, standard random matrix theory would not give the zeroth order statistical expectation, but the localization would still be determined by the return dynamics in the semiclassical approximation; see [9] for the organizational scheme of the return dynamics in integrable systems.

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