Global Instability in Fully Nonlinear Systems

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Existence of a saturated steady solution of a nonlinear evolution equation subject to a boundary condition at $x = 0$, called a nonlinear global mode, is illustrated on the real subcritical Ginzburg-Landau model. Such a nonlinear global mode is shown to exist whereas the flow is linearly stable, convectively unstable, or absolutely unstable. If the linearized evolution operator is absolutely unstable, then a global mode exists but the converse is false. This result relies only on the existence of a structurally unstable heteroclinic orbit in the phase space and is likely to be generic as demonstrated by the supercritical Ginzburg-Landau and the van der Pol-Duffing equations. [S0031-9007(96)01599-2]

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Wakes behind bluff bodies [1], mixing layers with back flow [2], and helium or heated jets [3] constitute examples of open flows where self-sustained oscillations at a specific intrinsic frequency develop when varying the control parameter (the Reynolds number, the velocity ratio, or the density ratio, respectively) and give rise to saturated amplitude states. In such open flows, Galilean invariance is broken by the boundary conditions (the body, the splitter plate, the nozzle, respectively) and the effect of the advection velocity cannot be removed. Therefore, one has to deal, not only with the growth of disturbances, but also with their propagation. This leads us to distinguish between absolute (A) and convective (C) linearly unstable flows. These concepts have been initially developed in the context of plasma physics [4]. They refer to the asymptotic behavior of the impulse response of the flow in the frame singled out by the boundary condition (hereafter named "laboratory frame"): a system is said to be linearly stable (S) when its linear response to an initial localized impulse decays asymptotically in any moving frame, and linearly unstable otherwise; it is absolutely unstable (A) if, at any fixed location, the response grows in time and convectively unstable if it decays (C). The exponential growth of initial disturbances when the flow is absolutely unstable will ultimately be compensated by nonlinearities to give rise to an intrinsic self-sustained resonance. On the other hand, a convectively unstable flow is supposed to relax to the basic flow but will strongly respond to any continuously applied forcing and therefore behaves as a spatial amplifier.

These concepts apply as soon as instability waves propagate in the laboratory frame. They have been successfully used to interpret the dynamics of binary convection [5], lasers [6], and the dynamo theory of disklike objects [7].

Recently, one of us proposed to extend these ideas to take into account nonlinearity by simply replacing the impulse linear wave packet, used in the standard definition, by any saturated wave packet of finite size in an infinite domain [8]. He shows that, for nonlinearly unstable systems, distinction between nonlinearly convective or absolute instability depends only on whether the trailing front, separating the basic state from a bifurcated state, moves downstream or upstream in the laboratory frame. The determination of the nonlinear absolute or nonlinear convective nature of the instability is therefore straightforward once the front velocity is determined [9].

But as noticed in Ref. [8], this definition based on front velocity is quite formal as the laboratory frame is artificially defined and no specific physical behavior is associated with one or the other type of instability.

When the dynamics of real open flows is of interest, one should take into account "entrance" conditions at the origin of the domain and determine the nonlinear global (NG) instability. This new concept is more physical as it refers to the existence of nonlinear solutions in semi-infinite domains with a boundary condition at $x = 0$. Since the laboratory frame is singled out, the advection velocity is now externally imposed and the selection problem, encountered to determine the front speed in an infinite domain, is replaced by the simpler problem of existence of a solution in $x \in [0, +\infty)$. As we shall demonstrate elsewhere [10], properties of the phase space which determine the occurrence of NG instability and those which prelude the front selection are remarkably connected although the physical considerations leading to one or the other are different. In the following, we focus on the determination of the NG region and its links to A and C. For clarity, it is worth insisting that the term "nonlinear global mode" refers to the solution of a nonlinear homogeneous eigenproblem involving the whole streamwise domain and therefore corresponds to the dynamical system terminology. In the global mode literature [11], the "linear global mode" term refers to the solution of an eigenproblem which also involves the whole streamwise domain but which is nonhomogeneous and linear. These notions of global and NG modes should overlap when nonlinear effects will be considered on the linear global mode dynamics and when inhomogeneous

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(nonparallel) effects will be taken into account in the structure of the presently studied nonlinear global mode.

The purpose of this Letter is to illustrate the relationship between NG and A/C instability using a one dimensional real Ginzburg-Landau equation in a semi-infinite domain which accounts for an extended subcritical pitchfork bifurcation:

$$\frac{\partial A}{\partial t} + U_0 \frac{\partial A}{\partial x} = \frac{\partial^2 A}{\partial x^2} + \mu A + A^3 - A^5, \qquad (1)$$

with the entrance condition $A(0) = 0$. Equation (1) depends on two control parameters (μ, U_0) independent of x. $U_0 \partial A/\partial x$ represents the effect of advection at the velocity $U_0 > 0$. Because of the symmetry $A \rightarrow -A$, only half of the solutions will be described. If the domain is made doubly infinite and the boundary condition $A(0) = 0$ is dropped, U_0 may be set to zero and classical results [12] are recovered: for $\mu < 0$, two linearly stable steady positive uniform states exist: $A_0 \equiv 0$ and $A_2 \equiv (1/2 + \sqrt{\mu + 1/4})^{1/2}$; for $\mu > 0$, A_0 becomes linearly unstable and A_2 remains the only stable state. Moreover, A_0 is linearly convectively unstable [13] (C) for $0 < \mu < U_0^2/4$ and linearly absolutely unstable (A) for $\mu > U_0^2/4$.

A nonlinear global mode is defined as a steady (or more generally an oscillatory) solution of (1) subject to the "entrance condition" (3) which therefore satisfies

$$U_0 dA/dx = d^2A/dx^2 + \mu A + A^3 - A^5, \qquad (2)$$

$$A(0) = 0\,, \qquad (3)$$

$$A(+\infty) = A_2\,. \qquad (4)$$

As discussed in [8], this asymptotic behavior (4) is imposed by the existence of a Lyapunov functional with a minimum in A_2. In the phase space $(A, dA/dx)$, each trajectory of (2) not ending at A_2 or A_0 is associated with an infinite Lyapunov energy and hence is not physical. Conditions (3) and (4) imply that a NG mode is represented by a trajectory linking a point where $A = 0$ but with $dA/dx \neq 0$, to A_2. A solution linking $(A = 0, dA/dx = 0)$ to A_2 is not a global mode because it corresponds to a front solution with $A(-\infty) = 0$ which cannot be renormalized in $A(0) = 0$.

For convenience, let us consider that the advection velocity U_0 is fixed and that we study the topological changes in the phase portrait while increasing the control parameter μ. For a fixed advection velocity, let us define $\mu_A(U_0)$ as the threshold of NG instability. Using a perturbative asymptotic expansion, we have rigorously determined the onset of a global mode which corresponds to the emergence of an intersection between the stable manifold of A_2 with the dA/dx axis at finite dA/dx:

$$\mu_A(U_0) = 3/16(U_0^2 + 2U_0/\sqrt{3} - 1) \quad \text{for} \quad U_0 < \sqrt{3}\,, \qquad (5)$$

$$\mu_A(U_0) = U_0^2/4 \quad \text{for} \quad U_0 > \sqrt{3}\,. \qquad (6)$$

For simplicity, only the topological demonstration on which the rigorous proof is based is given here. As shown graphically in [8] for the case $\mu_A < 0$ and in Fig. 1 for the other cases, the emergence of an intersection between the stable manifold of A_2 and the dA/dx axis implies the existence at threshold of a structurally unstable heteroclinic trajectory between A_0 and A_2. When $U_0 < 1/\sqrt{3}$, [8] has shown the structure of the heteroclinic orbit and its perturbations. In this case, $\mu_A(U_0)$ is negative; therefore, A_0 is a saddle and a heteroclinic orbit connecting A_0 to A_2 exists for a single value of $\mu = \mu_A(U_0)$. This orbit is structurally unstable as it departs from A_0 along the unstable eigendirection. Its perturbation for $\mu > \mu_A(U_0)$ give rise to an intersection between the stable manifold of A_2 and the axis dA/dx

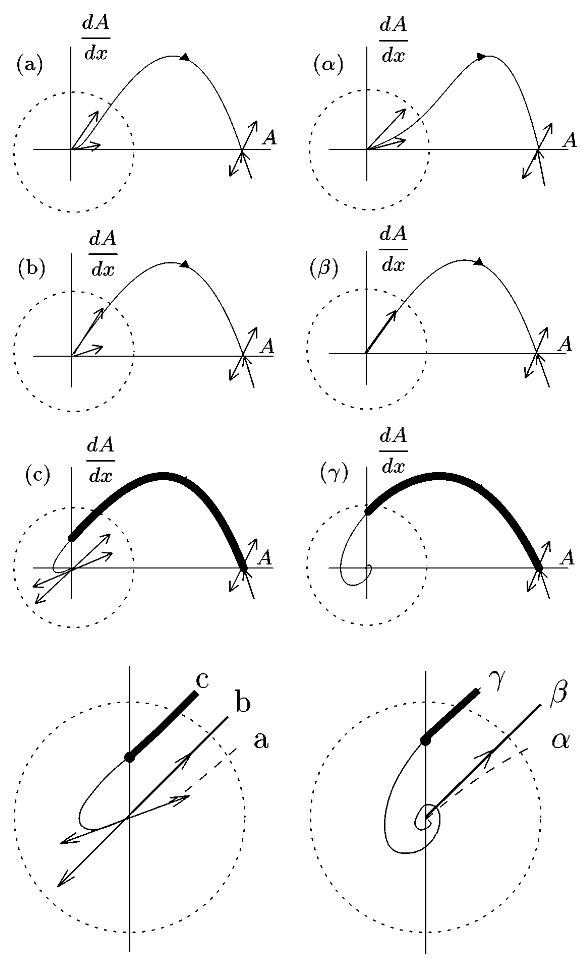

FIG. 1. Topological changes in the structure of the stable manifold of A_2 leading to the NG instability when keeping U_0 constant and increasing μ through the NG threshold $\mu_A(U_0)$. Left column: "global" instability case $1/\sqrt{3} < U_0 < \sqrt{3}$; (a) $0 < \mu < \mu_A$; (b) $\mu = \mu_A$; (c) $\mu > \mu_A$; right column: "local" instability case $U_0 > \sqrt{3}$; (α) $0 < \mu < \mu_A$; (β) $\mu = \mu_A$; (γ) $\mu > \mu_A$. The solution drawn by a *continuous heavy line* is the NG mode. In each case, the bottom figure presents the close up of the modifications which take place around A_0, a, b, c, α, β, γ referring to the labels in the figure.

(see Ref. [8]). When $U_0 > 1/\sqrt{3}$, the problem is less simple since the transition occurs for $\mu_A(U_0) > 0$, while A_0 is an unstable node. As shown in Fig. 1 where all possible cases obtained by numerical solutions of (2)–(4) are reproduced, whatever the value of μ positive, a heteroclinic orbit connecting A_0 to A_2 exists. For $1/\sqrt{3} < U_0 < \sqrt{3}$, and $\mu < \mu_A(U_0)$, the heteroclinic orbit always departs from A_0 to the right along its least unstable eigendirection (as it should generically [14]) [Fig. 1(a)]; at $\mu = \mu_A(U_0)$, an exceptional connection along the most unstable direction of A_0 appears [Fig. 1(b)]; when $\mu > \mu_A(U_0)$ the connection returns to the least unstable direction of A_0 [Fig. 1(c)] but from the other side of the dA/dx axis. In this case, the stable manifold of A_2 possesses a portion in $A < 0$ and therefore must cross the dA/dx axis giving rise to a NG mode plotted by a heavy line in Fig. 1(c). As for the former case ($U_0 < 1/\sqrt{3}$), an apparition of a global mode is linked to the crossing of a control parameter value for which a heteroclinic orbit between A_0 and A_2 exists and is structurally unstable because it departs from A_0 along its most unstable eigendirection. In this case, the transition is not associated to a local change around A_0 but to the global structure of the phase space and therefore cannot be detected by a linear analysis.

For $U_0 > \sqrt{3}$ and $\mu < \mu_A(U_0)$, the heteroclinic orbit always departs from A_0 to the right along its least unstable eigendirection [Fig. 1(α)] but for $\mu = \mu_A(U_0)$, the two eigendirections coincide [Fig. 1(β)]. For $\mu > \mu_A(U_0)$, the heteroclinic orbit spirals out of A_0 to A_2 [Fig. 1(γ)] giving rise to infinitely many global modes, only the first one being represented by heavy lines in Fig. 1(γ)]. At the bifurcation, the heteroclinic orbit is therefore structurally unstable but this time "locally" because the nature of the A_0 fixed point changes from node to focus. This local change occurs when two eigendirections near A_0 coincide, i.e., when the linear instability changes from linear convective to absolute (at $\mu = U_0^2/4$). Following the value of U_0, the threshold $\mu_A(U_0)$ has been determined by the global or local structural instabilities, depending which one occurs first. Only in the latter case does the NG threshold coincide with the A threshold.

These results have been demonstrated analytically by seeking the heteroclinic trajectory linking A_0 and A_2 as a polynomial or a series expansion in A. For each system, the existence of a NG mode for $\mu > \mu_A$ is then rigorously proved by matched asymptotic expansions with an inner linear region close to A_0 and a nonlinear region outside (the principle sketched in Fig. 1).

These results are synthesized in Fig. 2: In the shaded region (light or dark), the system is NG unstable. The dark shading corresponds to the A region whereas the light shaded region corresponds to a NG bifurcation which occurs while the basic state A_0 is linearly stable $\mu < 0$ or linearly convectively unstable $\mu > 0$. As we have shown, the NG threshold precedes the A threshold as soon as a structurally unstable heteroclinic orbit appears in the phase space. This appearance involves the nonlinear solution of the system along the whole x domain and therefore cannot be obtained from a linearized approach. For $U_0 < \sqrt{3}$, the NG threshold (heavy line) always precedes the A threshold (discontinuous line) whereas for $U_0 > \sqrt{3}$, the system becomes NG unstable and A unstable simultaneously.

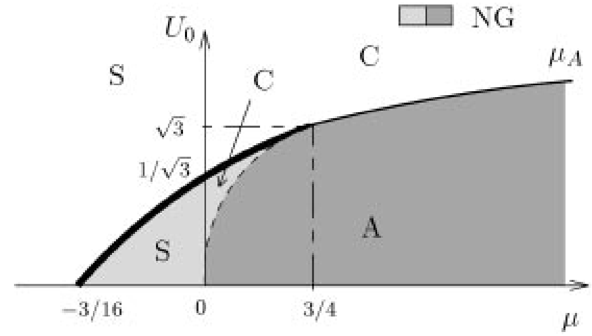

FIG. 2. NG domain in parameter space for the model (1). The dark gray region represents the absolutely (A) unstable region which is totally embedded in the NG region. In the light gray region, a NG exists whereas the system is not absolutely unstable. In this case, resonances are not predicted from a linear analysis but from a fully nonlinear one.

This result is likely to be general and does not seem to depend on the particular choice of a model equation. In the classical supercritical model

$$\frac{\partial A}{\partial t} + U_0 \frac{\partial A}{\partial x} = \frac{\partial^2 A}{\partial x^2} + \mu A - A^3, \qquad (7)$$

only one parameter μ or U_0 is necessary to describe the system, since one of them can be set to unity by rescaling A, x, and t. As a result, only one kind of transition should be observed by lack of degrees of freedom; we have found that the system becomes NG unstable and A unstable at the same time for $\mu_A = U_0^2/4$. But as soon as an extraterm is added to (7), for example, a nonlinear contribution to the advection [Eq. (8)], the NG region starts extending beyond the A domain. This system is known as the van der Pol-Duffing model:

$$\frac{\partial A}{\partial t} + (U_0 - A^2)\frac{\partial A}{\partial x} = \frac{\partial^2 A}{\partial x^2} + \mu A - A^3. \qquad (8)$$

The region of NG instability for model (8) in a semi-infinite domain with the condition (3) is bounded by the curves $\mu_A(U_0) = U_0^2/4$ for $U_0 < 6$, $\mu_A(U_0) = 3U_0 - 9$ for $U_0 > 6$. By contrast with model (1), the system becomes NG unstable and A unstable at the same value of the bifurcation parameter only for $U_0 < 6$, i.e., for small advection velocities. For $U_0 > 6$, the system is NG while being C unstable showing that this property arises even in the supercritical bifurcation case.

As already noted, the threshold of NG instability, which has been rigorously linked to the existence of a structurally unstable heteroclinic orbit linking A_0 to A_2, is closely related to the selection of front velocity in infinite domains. van Saarloos and Hohenberg [9] have recently established on physical grounds a selection principle for the front velocity in the complex Ginzburg-Landau equation, which is in fact similar to our criterion for the appearance of a global mode. We are now able to propose an alternative interpretation of their results by conjecturing that in an infinite domain, the velocity v_f of the selected front between the basic state to the left and the bifurcated state to the right will be such that in the semi-infinite domain, a NG mode exists for advection velocities $U_0 < -v_f$ (v_f is counted positive to the right). If $-v_f$ is larger than U_0, a front initially imposed far downstream in a semi-infinite domain will move upstream without "feeling" at first the boundary condition at $x = 0$ and stop close to the boundary where it is "caught" at $x = 0$.

In conclusion, we have presented results concerning the nonlinear global (NG) instability vs the absolute (A) or convective (C) nature of the instability of a basic state using very simple models. NG are nonzero steady (or oscillatory) solutions of a well-posed, nonlinear system with a boundary condition at the origin. The existence of such a solution, which depends not only on the instability parameter, but also on the advection velocity, has been determined rigorously by matched asymptotic expansions. This solution appears while increasing μ as soon as a heteroclinic orbit linking the basic solution to the bifurcated solution in the phase space $(A, dA/dx)$ is structurally unstable. This structural instability may be either local, because it involves a change in the nature of the basic fixed point A_0 (the transition then corresponds to the linear absolute instability threshold), or global because the stable manifold of A_2 connects to the most unstable direction of A_0. In the latter case, the NG transition occurs while the system is either locally stable or locally convectively unstable. This implies immediately that a system is always NG unstable before being A unstable. We conjecture that the precedence of the NG instability over the A instability is general and therefore holds for any kind of instability in fluid mechanics, in plasma physics, in chemical reactions, or in nonlinear optics. An experimental confirmation of these results could be attempted by adding a mean flow to a chemical reaction. For example, Hanna *et al.* have studied propagating fronts in the iodate oxidation of arsenous acid [15] and have computed the velocity of a front appearing as a narrow blue band propagating through a colorless solution. This velocity seems nonlinearly selected. By rendering their experimental system open and setting an advection velocity in the solution, the emergence of NG modes should be observed as a steady blue narrow band surrounded by a colorless solution which, following the advection velocity magnitude, may happen while the solution is stable or convectively unstable.

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