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Asymptotic Behavior of N-Soliton Trains of the Nonlinear Schrödinger Equation

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The generalization of the adiabatic perturbation approach of N-soliton interactions in the nonlinear Schrödinger equation (NLSE) has been shown to lead to the complex Toda lattice with N nodes. This allows us to predict the asymptotic behavior of trains of N solitonlike pulses with approximately equal amplitudes and velocities, but with arbitrary phase differences. These predictions agree very well with the numerical results. [S0031-9007(96)01454-8]

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It is well known that the nonlinear Schrödinger equation and its perturbed version,

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u(x,t) = iR[u], \qquad (1)$$

model a number of physical phenomena [1–10], including nonlinear beam propagation in the refractive media [9] and optical pulse propagation in fibers [5–11]. In what follows we will use the language of the temporal solitons from the fiber optics, but these results are valid also for the spatial solitons in nonlinear refractive media [9]. Of importance here is the stability of a train of soliton pulses. The question is how close can one pack such a train and then be able to stably propagate it over a given distance? We present an analytical model wherein one can study this stability for various combinations of amplitudes, phases, and spacings.

Soliton evolution, when $R[u] = 0$, with the solitons having different velocities, is also well known, see, e.g., [3,4]. However, when the solitons have (nearly) the same velocities, if one attempted to use the same analytical methods [12–14], he would be faced with the complexity of exact N-soliton solutions and eigenfunctions. A better method would be to recognize that the only N-soliton solution of interest is where the solitons are spatially separated. Perturbative methods do exist for such soliton solutions. They are applicable when each soliton can be treated as a separate entity (particle) and when the effect of the interaction and (or) perturbation is a slow deformation of the soliton parameters (see [5,14–17] and the numerous references therein). Most of the results in that direction have been devoted to one-soliton [18] and two-soliton interactions [17] taking into account several types of perturbations [7,15,16,19]. The two-soliton case with $R[u] = 0$ has been solved analytically by Karpman and Solov'ev [17]. For the other cases, there are numerous numerical investigations; see [11,15,20,21].

Very few analytical results are known for the general case of N-soliton interaction with $N \geq 3$. Here we refer to the papers by Gorshkov and Arnold [22,23] who conjectured that the soliton positions for an infinite train of solitonlike pulses should obey the Toda chain equation.

Recently in [24–26] there was derived a natural generalization of the Karpman-Solov'ev's equations for N solitons. Next it was shown [26,27] that after some additional approximations the corresponding dynamical system of $4N$ equations for the soliton parameters [see Eqs. (2)–(5) below] simplifies to the complex Toda chain equations (CTC) with N nodes, which is the natural generalization of Gorshkov-Arnold results.

Here we use the well known facts about the Toda chain [28] which can be generalized also to the complex case: (1) The CTC allows Lax representation with a Lax matrix L. (2) It has N (complex) integrals of motion in involution, which are provided by the eigenvalues ζ_k of L. (3) The leading terms of the asymptotics of the solutions for $t \to \infty$ are determined by ζ_k (see also the second remark below. (4) The eigenvalues are determined by the initial conditions.

Thus to determine the asymptotics of the CTC, we only need to know the initial values of the soliton parameters. Consequently we can evaluate the eigenvalues of L at $t = 0$ and then use them to predict the asymptotic of the soliton parameters (positions, phases, and velocities) for $t \to \infty$.

Such an approach is justified when we consider a special class of initial conditions: equidistant and well separated solitonlike pulses with (nearly) the same amplitudes and velocities. We obtain an analytical formula for the asymptotics of the soliton parameters as a function of the initial distance, r_0, and phase difference, δ_0, between the adjacent solitons. (We assume that these are the same between any adjacent pair of solitons.) For a small parameter, we use $\epsilon = e^{-2\nu_0 r_0}$, which comes from the overlap between the neighboring solitons at $t = 0$ and determines the slow-time scale. Then we find that the asymptotic velocity of the kth soliton is of the order of $\sqrt{\epsilon}\cos(\pi k/(N+1))$, where N is the total number of soliton pulses. We have compared our results for the soliton velocities against numerical results and find them to be in good agreement (to within a few percent).

Obviously, if the solitons separate, they will evolve as free particles. The fact that this is possible for certain initial conditions described in the next section has been established by numeric simulations in [20]. The present paper proposes an analytic explanation of this fact.

Finally, our model can be modified to take into account various perturbations on the right-hand side of (1), which will then lead eventually to perturbed versions of the CTC.

N-soliton interactions and CTC.—The generalization of the Karpman-Solov'ev approach to the case of $N > 2$ solitons in the lowest order leads to the following system of $4N$ dynamical equations for the soliton parameters [24–26];

$$\frac{d\nu_k}{dt} = 16\nu_k^2(S_{k,k-1} - S_{k,k+1}), \tag{2}$$

$$\frac{d\mu_k}{dt} = -16\nu_k^2(C_{k,k-1} - C_{k,k+1}), \tag{3}$$

$$\frac{d\xi_k}{dt} = 2\mu_k - 4(S_{k,k-1} - S_{k,k+1}), \tag{4}$$

$$\begin{aligned}\frac{d\delta_k}{dt} &= 2(\mu_k^2 + \nu_k^2) - 8\mu_k(S_{k,k-1} + S_{k,k+1}) \\ &\quad + 24\nu_k(C_{k,k-1} + C_{k,k+1}),\end{aligned} \tag{5}$$

where μ_k, ν_k, ξ_k, and δ_k are the velocity, amplitude, position, and phase of the kth soliton pulse and

$$S_{k,n} = e^{-|\beta_{kn}|}\nu_n \sin s_{k,n}\phi_{kn}, \tag{6a}$$

$$C_{k,n} = e^{-|\beta_{kn}|}\nu_n \cos\phi_{kn}, \tag{6b}$$

$$\beta_{kn} = 2\nu_n(\xi_k - \xi_n), \tag{6c}$$

$$\phi_{kn} = \delta_k - \delta_n - 2\mu_n(\xi_k - \xi_n), \tag{6d}$$

and $s_{k,n} = \operatorname{sign}\beta_{kn}$. We assume, without loss of generality, that $\xi_k < \xi_{k+1}$. This system is valid provided the soliton parameters satisfy $|\mu_k - \mu_n| \ll \mu, |\nu_k - \nu_n| \ll \nu$, $\nu|\xi_{0k} - \xi_{0n}| \gg 1$, and $|\nu_k - \nu_n||\xi_{0k} - \xi_{0n}| \ll 1$, where ν and μ are the average amplitude and velocity. They ensure that the N-soliton solution can be well approximated by the sum of N one soliton terms.

We are interested in simplifying and solving the system (2)–(5) in the limit of $\nu r_0 \gg 1$. In this limit we have two time scales. The first is the fast time of order unity, over which the phase and position changes. The second is the slow time, of order $e^{-\nu_0 r_0}$, over which the action variables change. If initially all the action variables are essentially equal, or $|\mu_k - \mu_n| \ll \mu, |\nu_k - \nu_n| \ll \nu$, then it follows that one may replace μ_k and ν_k in S_{kn} and C_{kn} by their average. In this case, one may ignore the slow-time variations in the action variables. Besides, the exponentially small terms (containing C_{kn} and S_{kn}) in Eq. (4) and (5) can be neglected as compared to the self-interacting ones [27].

Now we define $C_{kn} - iS_{kn} \simeq e^{s_{kn}(q_k - q_n)}$, where $n = k \pm 1$, and $s_{k,k-1} = -s_{k,k+1} = 1$ with

$$\begin{aligned}q_{k+1} - q_k = &- 2\nu(\xi_{k+1} - \xi_k) + \ln 4\nu^2 \\ &+ i[\pi + 2\mu(\xi_{k+1} - \xi_k) - (\delta_{k+1} - \delta_k)].\end{aligned} \tag{7}$$

These notations are consistent up to terms of second order with respect to $(\nu_k - \nu)^2$ and $(\mu_k - \mu)^2$. Then the system (2)–(5) simplifies to the Toda chain system with N nodes for the complex functions q_k,

$$\frac{d^2 q_k}{d\tau^2} = e^{q_{k+1} - q_k} - e^{q_k - q_{k-1}}, \tag{8}$$

where $k = 1, \dots, N$ and $\tau = 4\nu t$; we also assumed that $e^{-q_0} = e^{q_{N+1}} = 0$.

Following Moser [28] we use the following Lax representation for (8):

$$\dot{L} = [B, L], \tag{9a}$$

$$L = \sum_{k=1}^{N}[b_k E_{kk} + a_k(E_{k,k+1} + E_{k-1,k})], \tag{9b}$$

$$B = \sum_{k=1}^{N} a_k(E_{k,k+1} - E_{k-1,k}). \tag{9c}$$

Here the matrices $(E_{kn})_{pq} = \delta_{kp}\delta_{nq}$, and $E_{kn} = 0$ whenever one of the indices becomes 0 or $N + 1$; the other notations in (9) are as follows:

$$a_k = \tfrac{1}{2}e^{(q_{k+1} - q_k)/2}, \tag{10a}$$

$$b_k = \tfrac{1}{2}(\mu_k + i\nu_k). \tag{10b}$$

The asymptotics of $q_k(t)$ for $t \to \infty$ will be given by [28]

$$q_k(t) = 8\nu\zeta_{N-k+1}t - B_k + \mathcal{O}(e^{-Dt}), \tag{11}$$

only if the eigenvalues ζ_k of L are such that

$$\operatorname{Re}\zeta_1 < \operatorname{Re}\zeta_2 < \cdots < \operatorname{Re}\zeta_N. \tag{12}$$

Then D in (11) is some real positive constant.

Then comparing (11) with (7) we get

$$\xi_{k+1} - \xi_k = 4(\zeta_{0,N-k+1} - \zeta_{0,N-k})t + \frac{1}{2\nu}(\ln 4\nu^2 + B_{0,k+1} - B_{0,k}) + \mathcal{O}(e^{-Dt}), \tag{13}$$

$$\delta_{k+1} - \delta_k = 8[\nu(\zeta_{1,N-k+1} - \zeta_{1,N-k}) + \mu(\zeta_{0,N-k+1} - \zeta_{0,N-k})]t + \pi + B_{1,k+1} - B_{1,k} + \frac{\mu}{\nu}(\ln 4\nu^2 + B_{0,k+1} - B_{0,k}) + \mathcal{O}(e^{-Dt}), \tag{14}$$

where $\zeta_k = \zeta_{0,k} + i\zeta_{1,k}$ and $B_k = B_{0k} + iB_{1k}$.

Analyzing the solution of the Toda chain provided by Moser [28] (see also [29]) one can derive explicit expressions also for the constants B_k in terms of the initial conditions.

Let us now evaluate the eigenvalues ζ_k of $L_0 = L|_{t=0}$ for the following choice in initial conditions:

$$\nu_k|_{t=0} = \nu, \tag{15a}$$

$$\mu_k|_{t=0} = 0, \tag{15b}$$

$$(\xi_{k+1} - \xi_k)|_{t=0} = r_0, \tag{15c}$$

$$(\delta_{k+1} - \delta_k)|_{t=0} = \delta_0. \tag{15d}$$

Then L_0, up to trivial factors, is related to the Cartan matrix of the algebra $sl(N)$ whose eigenvalues and their corresponding eigenvectors are known [30]. Thus the eigenvalues ζ_k are equal to

$$\zeta_k = \frac{i\nu}{2} + 2\nu e^{-\nu r_0}\left(\sin\frac{\delta_0}{2} + i\cos\frac{\delta_0}{2}\right)\cos\theta_{N-k+1}, \tag{16}$$

where $\theta_k = \pi k/N + 1$.

For the differences $B_{k+1} - B_k$ with the same initial conditions (15) we find

$$B_{k+1} - B_k = 2\nu r_0 - 2\ln\left(\frac{2\alpha_{k+1}\sin\theta_{k+1}}{\alpha_k\sin\theta_k}\right) - \ln 4\nu^2 + i(\delta_0 - \pi), \tag{17a}$$

where $\alpha_1 = 1$ and

$$\alpha_k = \prod_{j=1}^{k-1}(\cos\theta_j - \cos\theta_k). \tag{17b}$$

Note that $\cos\theta_j - \cos\theta_k > 0$ for all $1 \le j < k \le N$ and $\sin\theta_k > 0$ for all $1 \le k \le N$. Thus the argument of the logarithm in (17) is always positive.

Therefore, inserting (16) and (17) into (13) and (14) for the initial conditions (15) we find ($\epsilon = e^{-2\nu r_0}$)

$$\mu_{k+1}^{\rm as} - \mu_k^{\rm as} = \sqrt{\epsilon}\, w_k \sin\frac{\delta_0}{2}, \tag{18}$$

$$\xi_{k+1} - \xi_k = 2\sqrt{\epsilon}\, w_k \sin\frac{\delta_0}{2}t + r_0 - \frac{1}{\nu}\ln\left(\frac{2\alpha_{k+1}\sin\theta_{k+1}}{\alpha_k\sin\theta_k}\right) + \mathcal{O}(e^{-Dt}), \tag{19}$$

$$\delta_{k+1} - \delta_k = 4\nu\sqrt{\epsilon}\, w_k \cos\frac{\delta_0}{2}t + \delta_0 + \mathcal{O}(e^{-Dt}), \tag{20}$$

$$w_k = 4\nu(\cos\theta_k - \cos\theta_{k+1}). \tag{21}$$

Let us list our results for some specific values of N and choices for the initial condition ($\nu = 1/2$, $\mu = 0$, $\delta_0 = \pi$, $r_0 = 6$, and $r_0 = 8$), for which numeric simulations are available. Such initial conditions describe, for example, joint propagation in optical fibers of N equal and equidistant solitons with phase difference π between them. We collect the results for $M_{N,k} = 2(\mu_{k+1}^{\rm as} - \mu_k^{\rm as})$ and $K_{N,k} = 2\operatorname{Re}(B_{k+1} - B_k)$ and compare them with the results from the numeric simulations (see the Table I). The theoretical values are obtained from the formulas (16) and (17). For example, inserting in them $N = 4$, $\mu = 0$, and $\nu = 1/2$ we get

$$M_{4,3} = M_{4,1} = 2\sqrt{\epsilon}, \tag{22a}$$

$$M_{4,2} = 2(\sqrt{5} - 1)\sqrt{\epsilon}, \tag{22b}$$

$$K_{4,3} = K_{4,1} = r_0 - \ln\frac{5 + \sqrt{5}}{5 - \sqrt{5}}, \tag{22c}$$

$$K_{4,2} = r_0 - 2\ln(5 - \sqrt{5}). \tag{22d}$$

TABLE I. Predictions and experimental results for the asymptotic values of the soliton positions for $N = 2,3,4,5$ and $\nu = 1/2$, $\mu = 0$, $\delta_0 = \pi$, $r_0 = 6, 8$.

N	k	$M_{N,k}^{\rm th}$	$M_{N,k}^{\rm num}$	$K_{N,k}^{\rm th}$	$K_{N,k}^{\rm num}$
			$r_0 = 6$		
2	1	0.1991	0.203	4.61	4.66
3	2, 1	0.1408	0.146	4.61	4.61
4	3, 1	0.0996	0.099	5.04	5.05
	2	0.1231	0.137	3.97	3.91
5	4, 1	0.0729	0.067	5.53	5.60
	3, 2	0.0996	0.111	3.99	3.82
			$r_0 = 8$		
2	1	0.0732	0.0737	6.61	6.63
3	2, 1	0.0518	0.0523	6.61	6.63
4	3, 1	0.0367	0.0365	7.04	7.07
	2	0.0452	0.0468	5.97	5.89
5	4, 1	0.0268	0.0262	7.53	5.57
	3, 2	0.0366	0.0378	5.99	5.95

We finish with two remarks. The first one concerns the evaluation of the eigenvalues of L for generic initial conditions. Then, of course, we have no nice analytic expressions like (16). For $N \leq 4$ the order of its characteristic equation does not exceed 4, and it can be solved analytically for any initial conditions. For larger values of N one can always solve for ζ_k numerically.

Our second remark is that CTC has a much richer class of solutions as compared to the real Toda chain (RTC). Like in the real case, the eigenvalues ζ_k of the matrix L are pairwise different [28]. However, now ζ_k are complex, and while $\zeta_k \neq \zeta_m$, it may happen that $\mathrm{Re}\,\zeta_k = \mathrm{Re}\,\zeta_m$ [i.e., (12) can be violated] and $\mathrm{Im}\zeta_k \neq \mathrm{Im}\zeta_m$. In such cases the asymptotic behavior, of the corresponding CTC solutions may substantially differ from the ones of RTC. For example, for some initial conditions the solutions of the CTC may form bound states or even develop singularities. These problems deserve special attention and are out of the scope of the present paper.

In conclusion, an analytical formula for the asymptotic velocities of the N-soliton pulses trains has been found in the special and experimentally important case when the initially equidistant soliton pulses have (nearly) equal amplitudes, velocities, and phase differences. The numerical check shows that separation $r_0 = 6$ is at the lowest threshold from which on our model becomes valid; for $r_0 = 8$ the precision is much better.

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