Excitation of Small Quantum Systems by High-Frequency Fields

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The excitation by a high-frequency field of multilevel quantum systems with a slowly varying density of states is investigated. A general approach is presented for a large class of driven systems with a smooth evolution operator. The Floquet eigenstates are characterized semiclassically on several energy scales. On a small scale sharp quasiresonance are found, whose shape is *universal* and independent of the field parameters and the details of the system. These are used to construct an effective tight-binding equation for the large-scale amplitude superimposed on the quasiresonances. [S0031-9007(96)01515-3]

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The excitation of quantum systems by external fields is a fundamental problem in quantum mechanics. Any interaction between matter and electromagnetic fields is described, on the simplest level, by such a model. Given the system at some initial condition, one would like to characterize its energy absorption from the external field as a function of time. In some cases, simple quantum systems can be described effectively by one degree of freedom. One-dimensional (1D) driven systems have been studied widely within the field of "quantum chaos" [1,2], since they provide some of the simplest examples of quantum systems which are chaotic in the classical limit. The phenomenon of dynamical localization, where the classical energy absorption associated with chaotic motion is suppressed by quantum interference [3,4], has been of special interest (for review, see S. Fishman in [1]).

In this work, a *general* approach is presented for systems with a slowly varying density of states, driven by a high-frequency field with matrix elements that vary slowly with the quantum numbers of the unperturbed system. We study the quasienergy (Floquet) eigenstates, which are the stationary states of the time dependent system [5], in the unperturbed representation, and characterize their properties over several energy scales. On the smallest scale, we find that these states are composed of a ladder of sharp peaks, or "quasiresonances" (QR), which are related to quantum nearly resonant transitions between the energies of the undriven system. While the existence of such peaks is well known [6,7], here they are derived from a local exact solution, which enables the quantitative description of their location and shape. Surprisingly, the QR's turn out to have a universal shape, independent of field parameters and of the details of the system. In particular, their width is independent of the driving field parameters, a result which contrasts simple perturbation theory or a two-level approximation. The "universality class" for which this result holds is characterized by a slowly varying unperturbed density of states, and slowly varying semiclassical matrix elements of the driving in the unperturbed representation. The universality of the QR's is surprising since we find that for a wide class of chaotic systems the characteristic behavior of the eigenstates is different from that found for a (banded) random matrix of similar bandwidth [8]. On a larger energy scale, our local solution allows us to construct an effective equation for the envelope of amplitudes superimposed on the QR's, similar in nature to the one obtained assuming that the resonances are of zero width [9]. The form of the local solution is used explicitly to write a tight-binding equation on the lattice of QR peaks. The parameters of this equation are nonuniversal and are related to the unperturbed spectrum and the matrix elements of the perturbation [10].

For systems with a slowly varying density of states, a constant energy spacing is a good approximation on a local energy scale, and the deviations of the spectrum from harmonicity appear as an adiabatic change of this spacing with energy. This "adiabatic nonlinearity" of the spectrum is characteristic of many 1D systems, of which several examples are the hydrogenlike atom ([11], D. L. Shepelyansky in [1]), charge bubbles in liquid helium [9,12,13], and surface electrons in a 2D metal in a perpendicular magnetic field [14]. This work presents a *general* framework for this class of systems, from which some new results emerge and some known results for special cases are confirmed. We focus on a monochromatic driving, which is of high frequency compared to the typical frequencies of the system. Thus, many unperturbed levels participate in the excitation, and a two-level approximation is inadequate. The driving field is not assumed weak and we *avoid* using perturbation theory. The basic idea employed in this work is to solve the problem in a limited energy regime, where the locally defined energy spacing can be considered constant. This local solution relies on the exact quantum mechanical solution of an integrable model [15], and the slow variation of the driving field matrix elements with quantum number is used explicitly. We then account for the large energy scales by exploiting the adiabatic

dependence of parameters on energy. A similar idea was previously applied to the "bubble" model [16]. Here it is shown that the results are quite general (more details in [17]), and apply to models of experimental relevance.

Consider the following 1D Hamiltonian in action-angle variables (I, θ) of the bound system:

$$\mathcal{H} = \mathcal{H}_0(I) + kV(I)g(\theta)\cos(\Omega t). \quad (1)$$

It is assumed that $\mathcal{H}_0(I)$ and $V(I)$ are smooth functions of I, and that $g(\theta)$ has a Fourier expansion with smooth, slowly decaying components $G(m)$. Our objective is to calculate the Floquet operator, and to characterize its eigenstates in the unperturbed representation $|n\rangle$, where $I = n\hbar$. In the high-frequency regime, the field period is a relatively short time scale and thus the semiclassical approximation is expected to be very accurate. Using also a leading order approximation to the classical trajectories [18], the Floquet eigenvalue equation is

$$e^{-i(T/\hbar)\mathcal{H}_0(n\hbar)}e^{-i/\hbar A(I;\hat{\theta})}|\psi_\lambda\rangle = e^{-i\lambda T}|\psi_\lambda\rangle, \quad (2)$$

where $T = 2\pi/\Omega$ is the period of the external driving, λ is the quasienergy, and

$$A(I,\theta) = k\int_0^T V(I)g(\theta + \omega(I)t)\cos(\Omega t)\,dt. \quad (3)$$

(For details, see [17].) The physical reason why the Floquet operator can be factored into a product of two operators, as in Eq. (2), is that most of the energy transfer takes place around the singular point of the potential. When approximating a 3D potential by one degree of freedom, usually one is dealing with the half line so a special point naturally arises. For the hydrogen atom, e.g., this is the nucleus; this has been the basis for constructing the Kepler map [19,20]. The existence of a singular point in space is related to the slow decrease of the Fourier components of the driving, $G(m)$.

We define a dimensionless parameter $\epsilon = \omega/\Omega$, where $\omega = \partial\mathcal{H}_0/\partial I$. In the absence of the driving, the solutions of Eq. (2) are δ functions in n space. For $\epsilon \ll 1$, the perturbation couples most effectively δ functions separated by approximately $j\hbar\Omega$ in energy (j integer), corresponding to nearly degenerate λ's. These form the QR ladder, which will be derived in what follows. For $\epsilon \ll 1$,

$$A(I;\theta) = \begin{cases} 0 & 0 < \theta < 2\pi(1-\epsilon), \\ R\cos\left[\frac{2\pi-\theta}{\epsilon}\right] & 2\pi(1-\epsilon) < \theta < 2\pi, \end{cases} \quad (4)$$

where $R = \frac{2\pi}{\omega}kV(I)G(m_+)$, and m_+ is the integer closest to $1/\epsilon$, corresponding to classical resonance $\Omega = m_+\omega$. The Fourier series of A is dominated by this resonance; the smoothness of the driving coefficients allows us to take into account not only this term but also its vicinity.

Since $\mathcal{H}_0(n\hbar)$ varies slowly as a function of n, it can be expanded around a large value n_0 to first order: $\mathcal{H}_0(n\hbar) \simeq \mathcal{H}_0(n_0\hbar) + \omega l\hbar$, where $l \equiv (n - n_0)$. Then the eigenvalue equation (2) appears as the equation for a "linear kicked rotor" [15], which is an exactly solvable model. Denoting the quasienergies of the linearized model by $\lambda^{\rm lin}$, its spectrum is $\lambda^{\rm lin}T \equiv \mu\omega T (\mathrm{mod}\ 2\pi)$, and the corresponding eigenstates are $|u\rangle e^{-i\lambda^{\rm lin}t}$, where

$$\langle l\,|\,u\rangle = e^{i\varphi}\sum_{m=-\infty}^{\infty} J_m(B)\,\mathrm{sinc}[\pi(\mu - l + m/\epsilon)] \quad (5)$$

with $\mathrm{sinc}(x) = \sin(x)/x$, $\varphi = \pi(\mu - l)(1 - 2\epsilon)$, and $B = (\pi/\hbar\omega)kV(n_0\hbar)G(m_+)/\sin(\pi/\epsilon)$. In the high-frequency limit, $\epsilon \ll 1$, the function (5) is composed of a chain of peaks separated by approximately the energy of one photon, each weighted by an amplitude. These QR are described by the function

$$Q_j(l) = \frac{\sin[\pi(l - l_j + \delta j)]}{\pi(l - l_j + \delta_j)}, \quad (6)$$

where l_j is the center of the QR and δ_j characterizes its precise shape. For a given quasienergy, the corresponding eigenstate (5) is composed of only a small fraction of all possible Q_j's, at positions determined by $\lambda^{\rm lin}$ through the relation $E_{l_j} = \lambda^{\rm lin}\hbar + j\hbar\Omega - \hbar\omega\delta_j$, with $|\delta_j| < 1/2$. Assuming that at each point tails of sinc functions centered far away contribute incoherently, the absolute value of the wave function is large on this sparse ladder of QR's. The width of the QR is independent of the driving field strength and of the density of unperturbed states. This is in contrast to the width associated with the transition rate given by Fermi's golden rule, and also in contrast to the Rabi width, both invalid approximations in our regime of parameters. It differs also from results found for systems modeled by random matrices [8].

The next step is to construct the eigenstates of the original nonlinear system by matching different energy regimes, where in each a local solution holds. In this construction, there are two important scales in action space. The first is the distance between classical resonances: our local solution is valid only outside the close vicinity of classical resonances. The second is the width of the resulting local solution, which is determined by the Bessel function in Eq. (5). Once the linearized local functions become wide enough to cover more than one classical resonance, it is expected that the different local solutions may interact and matching between the regions is required. It turns out that for the two classes of models described below, this matching condition coincides with the Chirikov criterion for resonance overlap [21]. This implies that for these cases, the matching should be imposed inside classically chaotic connected regions of phase space. Outside such regions, the decay of the eigenstates is expected to be determined by classical bounds. The QR's (6) are narrow compared to the classical structures; therefore their shape, obtained from the solution of the linearized equation, is not expected to be strongly effected by these structures, which vary only on much

larger scales. The long range behavior of the linearized equation, determined by the Bessel function in (5), does not hold for the full nonlinear problem and therefore the matching outlined above is required.

Under conditions that the matching is valid, we use

$$E_{n_j} \approx E_{n_0} + (\partial E_n/\partial n)_{n_0} l_j \tag{7}$$

to find that in the matched nonlinear solution, the QR appear on the ladder of states n_j satisfying

$$E_{n_j} = \lambda\hbar + j\hbar\Omega - (\partial E_n/\partial n)_{n_0}\delta_j\,. \tag{8}$$

Thus the eigenvalue λ sets the origin of the ladder for the corresponding eigenstate, and the QR's are numbered by their position j on this ladder. Each is described approximately by the function $Q_j(n)$ of Eq. (6), characterized by the peak position n_j and the detuning $|\delta_j| < 1/2$, both satisfying Eq. (8). The matching may alter the amplitudes of the QR's; therefore we write the global eigenstate in the following approximate form:

$$\langle n \,|\, u\rangle = \sum_j A_j Q_j(n) = \sum_j A_j \frac{\sin[\pi(n - n_j + \delta_j)]}{\pi(n - n_j + \delta_j)}\,. \tag{9}$$

This equation, together with Eq. (8) which defines the parameters n_j and δ_j, constitutes the main result concerning the structure of the Floquet eigenstates on a small scale. They predict, for a given λ, the precise location and shape of the QR's of the corresponding eigenstate.

We now turn to determine the amplitudes A_j. The Floquet eigenvalue equation can be equivalently written in an extended Hilbert space of variables (n, j) of which $|n, j\rangle = |n\rangle e^{-ij\Omega t}$ are basis functions [22]:

$$(E_n - \hbar j\Omega)\phi_{n,j} + \tfrac{k}{2}\Sigma_{n'}\langle n|\hat{\mathcal{O}}|n'\rangle[\phi_{n',j+1} + \phi_{n',j-1}] = \hbar\lambda\phi_{n,j}\,, \tag{10}$$

where $\phi_{n,j} = \langle n, j \,|\, \phi\rangle$ and $\mathcal{O} = V(I)g(\theta)$. For smooth functions $V(I)$ the semiclassical matrix elements may be written as $\langle n|\hat{\mathcal{O}}|n'\rangle \approx V(I)G(n - n')$. Using the local structure of the QR's and the properties of the sinc functions on a lattice, the equation reads

$$\frac{2\hbar\omega}{\pi k V(\hbar n_j)}\sin(\pi\delta_j)A_j - G^+A_{j+1} - G^-A_{j-1} \approx 0\,, \tag{11}$$

where $G^\pm = G[(\mp 1/e_{j\pm1}) + \delta_{j\pm1}]$ and $\epsilon_{j\pm1}$ is the value of ϵ at the $(j \pm 1)$th QR. This is a tight-binding equation on the 1D lattice of QR's labeled j. The diagonal potential is determined by the δ_j, which are the detunings of the energies E_{n_j} from exact resonance, and the hopping is related to matrix elements of the driving between neighboring QR's. The fact that there is only near-neighbor coupling is a result of the driving being harmonic.

We consider two classes of models in some detail. These are described by the Hamiltonian

$$\mathcal{H}(p, x) = \frac{p^2}{2m} + bx^\sigma + k\hat{\mathcal{O}}\cos(\Omega t), \qquad x \geq 0\,, \tag{12}$$

where σ is a real number, $-2 < \sigma < 2$ ($b < 0$ for $\sigma < 0$). The driving field is described in the dipole approximation, and it is convenient to use different gauges for the cases of positive or negative σ; thus $\hat{\mathcal{O}} = \hat{x}$ for $\sigma > 0$, and $\hat{\mathcal{O}} = \hat{p}/\Omega$ for $\sigma < 0$. Both classes have a slowly varying density of states, the interaction with the field is described by a function $V(I)$ which is slowly varying with I, and $g(\theta)$ with smooth Fourier coefficients. For high energies, (12) is well approximated by (1). Thus, on the fine scale they exhibit the universal QR structure. The differences between the models appear in the envelope of amplitudes superimposed on the peak structure.

For $\sigma > 0$, the binding potentials have a trianglelike singularity near the origin $x = 0$. The special case of $\sigma = 1$ corresponds to the bubble model, which is a triangular potential well. The dipole matrix elements have the asymptotic form $\langle n|\hat{x}|m\rangle \propto (n\hbar)^{2/(2+\sigma)}/(n - m)^2$. The eigenstates for this case may be characterized by two qualitatively different regimes in n space. In the small n regime they are exponentially localized, while for larger values of n they are more extended, with a crossover point n_c between the two regions, satisfying

$$n_c \sim (\Omega^2/k)^{(2+\sigma)/\sigma}. \tag{13}$$

This is a generalization of a result previously found for the special case of the bubble model [16]. In the asymptotic regime $n \to \infty$, Eq. (11) becomes similar to the Anderson model in a static electric field:

$$\left(\frac{\sin(\pi\delta_j)}{\sqrt{j}}\right)\left(\frac{\sqrt{\hbar}\,\Omega^{3/2}}{C_1 k}\right)A_j + R^+A_{j+1} + R^-A_{j-1} \approx 0\,, \tag{14}$$

where C_1 is a constant and $R^\pm = (1 \mp \delta_{j\pm1}\epsilon_{j\pm1})^{-2}$. The diagonal potential is given by the pseudorandom function $\sin(\pi\delta_j)$, where $\delta_j = \mathrm{frac}\{\frac{1}{\hbar}[\hbar\alpha(\lambda + j)]^{1/\alpha}\}$ and $\alpha = 2\sigma/(2 + \sigma)$, damped along the lattice by $1/\sqrt{j}$. The hopping terms depend weakly on position j. For a random potential and strictly constant hopping, the eigenstates are power localized with a power proportional to the square of the prefactor of the diagonal potential [23]. Numerical calculations for the model (14) with $\sigma = 1$ provide evidence that these differences do not alter the qualitative conclusion from the theorem (see [16], following [9]). Thus the eigenstates are power-law decaying,

$$A_j \sim 1/j^{(c_1\hbar\Omega^3/k^2)}, \tag{15}$$

where c_1 is a constant. This implies the existence of a critical crossover field strength $k_c \sim \sqrt{\hbar\Omega^3}$ beyond which the tails of the eigenstates turn non-normalizable. A similar result was stated previously in [9]. For the special case of the bubble model, it was first obtained by

use of a Kepler-like map [13]. The reason that in this regime the result is independent of σ is related to the dipole matrix elements being asymptotically similar for all σ, which is a result of the trianglelike singularity that all these potentials have at the origin.

For $\sigma < 0$, the potential is singular at the origin ($\lim_{x\to 0} x^{\sigma} = -\infty$). The special case of $\sigma = -1$ corresponds to the 1D Coulomb potential, which is a good approximation for hydrogenlike Rydberg atoms with small angular momentum [24]. Using the asymptotic form $\langle n|\hat{p}|m\rangle \propto i\,\mathrm{sgn}(n-m)\,(n\hbar)^{\sigma/2+\sigma}/|n-m|^{2/2-\sigma}$, the resulting equation for the amplitudes is

$$\left(\frac{\hbar\Omega^{\eta}}{C_2 k}\right)\sin(\pi\delta_r)A_r + R^{+}A_{r+1} - R^{-}A_{r-1} \approx 0\,, \quad (16)$$

where C_2 is a constant, $R^{\pm} = (1 \mp \delta_{r\pm1}\epsilon_{r\pm1})^{2/(\sigma-2)}$, $\eta = (4-\sigma)/(2-\sigma)$, and $\delta_r = \mathrm{frac}\{\frac{1}{\hbar}[\alpha(\hbar\lambda - (r_{\max} - r)\hbar\Omega)]^{1/\alpha}\}$ for $r < r_{\max}$; since the bound part of the spectrum is of finite range in energy, there is only a finite number of QR's. For $\epsilon < 1$, the diagonal potential may be considered similar to a random one. Neglecting the weak dependence of the hopping terms on r, one obtains a 1D Anderson model for the amplitudes A_r. The eigenstates of this model are exponentially decaying with a perturbative estimate for the localization length given by $\xi \sim k^2/\hbar^2\Omega^{2\eta}$. Also in this case, numerical calculations confirm that the weak dependence of the hopping terms on position does not alter the existence of exponential localization and its scaling with parameters. Our description neglects effects of the continuum, and thus is valid only far away from it; the above expression for ξ is meaningful only if it is smaller than the lattice size, which is finite in this case.

In conclusion, for a particle in a 1D potential well, where the unperturbed spectrum is slowly varying, driven by a harmonic high-frequency field, some properties of the Floquet eigenstates were found on several energy scales. The small scale resonant structures, the QR's, are nonperturbative and *universal*, independent of the details of the system and on the parameters of the driving. Thus the general form of the eigenstate is described by Eqs. (8) and (9), and is very different from the one found for banded random matrices. On a larger scale, it was shown in general how to construct a tight-binding equation, Eq. (11), for the amplitudes A_j of (9) superimposed on the QR's. Two specific classes were considered, where this equation turns out to be an Anderson model on a finite lattice, and an Anderson model on an infinite lattice with a constant electric field. It will be of interest to study experimentally the nature of the eigenstates and in particular the shape of the quasiresonances for some systems that belong to classes explored in the present work. Recent developments in the optical trapping of atoms may enable such investigations for various potentials.

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