Global Persistence Exponent for Nonequilibrium Critical Dynamics

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(Received 17 June 1996)

A "persistence exponent" θ is defined for nonequilibrium critical phenomena. It describes the probability, $p(t) \sim t^{-\theta}$, that the global order parameter has not changed sign in the time interval t following a quench to the critical point from a disordered state. This exponent is calculated in mean-field theory, in the $n = \infty$ limit of the $O(n)$ model, to first order in $\epsilon = 4 - d$, and for the 1D Ising model. Numerical results are obtained for the 2D Ising model. We argue that θ is a new independent exponent. [S0031-9007(96)01451-2]

PACS numbers: 05.70.Ln, 05.50.+q, 05.70.Jk,

For many years it was believed that critical phenomena were characterized by a set of three critical exponents, comprising two independent static exponents (other static exponents being related to these by scaling laws) and the dynamical exponent z. Then, quite recently, it was discovered that there is another dynamical exponent, the "nonequilibrium" (or "short-time") exponent λ, needed to describe two-time correlations in a system relaxing to the critical state from a disordered initial condition [1,2]. It is natural to ask "Are there any more independent critical exponents?". In this Letter we propose such an exponent—the "persistence exponent" θ associated with the probability, $p(t) \sim t^{-\theta}$, that the global order parameter (e.g., the magnetization of a ferromagnet) has not changed sign in time t following a quench to the critical point from the high-temperature phase. We calculate θ in mean-field theory, in the $n = \infty$ limit of the $O(n)$ model, to first order in $\epsilon = 4 - d$ (d = dimension of space) and for the $d = 1$ Ising model. In fact, it turns out that all these results satisfy the scaling law $\theta z = \lambda - d + 1 - \eta/2$, which can be derived on the assumption that the dynamics of the global order parameter is a Markov process. We shall argue, however, that this process is in general non-Markovian, so that θ is in general a new, nontrivial critical exponent.

The persistence exponent θ was first introduced in the context of the nonequilibrium coarsening dynamics of systems at zero temperature [3,4]. In that context it describes the power-law decay, $p(t) \sim t^{-\theta}$, of the probability that the local order parameter $\phi(\mathbf{x})$ has not changed sign during the time interval t after the quench to $T = 0$. Equivalently, it gives the fraction of space in which the order parameter has not changed sign up to time t. More generally, one can consider the probability $p_0(t_1, t_2)$ of no sign changes between t_1 and t_2. Scaling considerations suggest $p_0(t_1, t_2) = f(t_1/t_2) \sim (t_1/t_2)^\theta$ for $t_2 \gg t_1$.

Exact solutions for one-dimensional systems [4,5] indicate that, in general, θ is a new nontrivial exponent for coarsening dynamics. Recently, we have shown that even the diffusion equation exhibits a nontrivial persistence exponent, and have developed a rather accurate approximate theory for this case [6]. The diffusion equation is itself a model of ordering dynamics, via the approximate theory of Ohta, Jasnow and Kawasaki (OJK) [7], and also describes, in its essential features, the ordering kinetics of the nonconserved $O(n)$ model in the large-n limit [8]: The exponents θ for these systems (OJK and large-n) are just those of the diffusion equation.

In this Letter we introduce and calculate the analogous exponent θ for nonequilibrium *critical* dynamics. In this case, however, one needs to consider the *global*, rather than the *local* order parameter. This is because individual degrees of freedom ("spins," say) are rapidly flipping so that the probability of not flipping in an interval t has an exponential tail. We shall see, however, that the probability for the *global* order parameter not to have flipped indeed decays as a power law. One simplifying property of the global order parameter is that, in the thermodynamic limit, it remains Gaussian at all finite times. This follows from the central limit theorem, on noting that the order-parameter field $\phi(\mathbf{x}, t)$ has a finite correlation length, $L(t) \sim t^{1/z}$. If the system has a volume $V \gg L(t)^d$, the appropriate Gaussian variable is the $k = 0$ Fourier component, $\tilde{\phi}_0(t) = [\int d^d x\, \phi(\mathbf{x}, t)]/\sqrt{V}$. From standard scaling, $\langle \tilde{\phi}_0^2(t) \rangle \sim L(t)^{2-\eta}$. This follows from the $k \to 0$ limit of the scaling form [1] $\langle \tilde{\phi}_{\mathbf{k}}(t) \tilde{\phi}_{-\mathbf{k}}(t) \rangle = k^{-(2-\eta)} G[kL(t)]$.

Our explicit results are derived from the Langevin equation for the vector order parameter $\vec{\phi} = (\phi_1, \ldots, \phi_n)$:

$$\partial_t \phi_i = \nabla^2 \phi_i - r\phi_i - (u/n)\phi_i \sum_j \phi_j^2 + \xi_i \,, \qquad (1)$$

where $\vec{\xi}(\mathbf{x}, t)$ is a Gaussian white noise with mean zero and correlator $\langle \xi_i(\mathbf{x}, t) \xi_j(\mathbf{x}', t') \rangle = 2\delta_{ij} \delta^d(\mathbf{x} - \mathbf{x}') \delta(t - t')$. [For a vector order parameter, we are defining $p(t)$ as

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the probability that a given *component* of the global order parameter, $\vec{\Phi}(t) = \int d^d x \vec{\phi}(\mathbf{x}, t)$, has not changed sign].

In mean-field theory, valid for $d \geq 4$, we set $r = 0 = u$. Then the $\mathbf{k} = 0$ Fourier component $\tilde{\phi}_i(0, t) = \Phi_i(t)/\sqrt{V}$ (where V is the volume) obeys the simple equation (suppressing the index i and the arguments)

$$\partial_t \tilde{\phi} = \tilde{\xi}\,, \tag{2}$$

indicating that $\tilde{\phi}$ executes a simple random walk. The nonflipping probability $p(t)$ is therefore just the probability that the random walker has not crossed the origin up to time t. It is given by [9] $p(t) \sim |\tilde{\phi}_0|/\sqrt{t}$ for large t, where $\tilde{\phi}_0$ is the initial value of $\tilde{\phi}$. Finally one has to average over $|\tilde{\phi}_0|$. For a disordered initial condition, $\Phi(0) \sim \sqrt{V}$ by the central limit theorem, so $\tilde{\phi}_0 = \Phi(0)/\sqrt{V}$ is $O(1)$, the desired average is also $O(1)$, and $p(t) \sim 1/\sqrt{t}$. We conclude that $\theta = 1/2$ in mean-field theory.

Next we consider the large-n limit. Equation (1) then simplifies to the self-consistent linear equation

$$\partial_t \phi = \nabla^2 \phi - (r + u\langle \phi^2 \rangle)\phi + \xi\,, \tag{3}$$

for each component. Defining $a(t) = -r - u\langle \phi^2 \rangle$ and $b(t) = \int_0^t a(t')\,dt'$, (3) has the Fourier-space solution

$$\tilde{\phi}(\mathbf{k}, t) = \tilde{\phi}(0, t)\exp[b(t) - k^2 t] + \int_0^t dt'\, \tilde{\xi}(\mathbf{k}, t')\exp[b(t) - b(t') - k^2(t - t')]\,. \tag{4}$$

It is easy to show that the second term, involving the noise, dominates the first at large t [1]. Retaining only the second, computing $\langle \phi^2 \rangle$, and defining $g = \exp(-2b)$ leads to the equation

$$\partial_t g = 2rg + 4u\int_0^t dt'\, g(t') \times \sum_{\mathbf{k}} \exp[-2k^2(t - t')]\,, \tag{5}$$

which can be solved by Laplace transformation. Setting r equal to its critical value, $r_c = -u\sum_{\mathbf{k}} k^{-2}$ gives

$$\bar{g}(s) = [s + 4u\{\bar{J}(0) - \bar{J}(s)\}]^{-1}, \tag{6}$$

$$\bar{J}(s) = \sum_{\mathbf{k}} (s + 2k^2)^{-1}, \tag{7}$$

from which one deduces $\bar{g} \sim s^{(2-d)/2}$ for $s \to 0$, for $2 < d < 4$. Inverting the Laplace transform gives (with $\epsilon = 4 - d$) $g(t) \sim t^{-\epsilon/2}$ for $t \to \infty$, whence $b \sim (\epsilon/4)\ln t$, and $a(t) \sim \epsilon/4t$.

The large-n equation of motion (3) therefore reduces to

$$\partial_t \tilde{\phi} = (\epsilon/4t)\tilde{\phi} + \tilde{\xi} \tag{8}$$

for the $\mathbf{k} = 0$ Fourier component of ϕ. The final step is to eliminate the first term on the right by setting $\tilde{\phi} = t^{\epsilon/4}\psi$, to give $\partial_t \psi = t^{-\epsilon/4}\tilde{\xi}(t)$. Introducing the new time variable $\tau = t^x$, this equation reduces to the random walk equation $\partial_\tau \psi = \eta(\tau)$, with η a Gaussian white noise, if one chooses $x = (2 - \epsilon)/2$. The final result is therefore $p(t) \sim \tau^{-1/2} = t^{(2-d)/4}$, giving

$$\theta = (d - 2)/4, \qquad 2 < d < 4 \ (n = \infty)\,. \tag{9}$$

For $d > 4$, θ sticks at the mean-field value of $1/2$.

Finally, we calculate θ to first order in $\epsilon = 4 - d$. This is most simply accomplished using the method of Wilson [10]. To order ϵ the calculation can be carried out in $d = 4$, by expanding $p(t)$ to first order in u, setting u equal to its renormalization-group (RG) fixed-point value, and exponentiating logarithms.

The perturbative calculation of $p(t)$ is in principle quite a difficult task. A systematic technique for performing the perturbation expansion was recently developed by two of us [11] in the general context of first-passage-time problems for non-Markov Gaussian processes. It amounts to expanding around the random walk (2) within a path-integral formulation of the problem. Since the global order-parameter $\Phi(t)$ remains Gaussian at all times (in the thermodynamic limit), this method is applicable. In the present work, however, we restrict ourselves to first order in ϵ, for which the result can be obtained by elementary methods. The reason is that the dynamics of $\Phi(t)$ remain Markov to this order, as we shall see.

First we replace u/n in (1) by u, to conform to the conventional notation for ϵ expansions. To first order in u, one can as usual replace the nonlinear term $u\phi_i \sum_j \phi_j^2$ in (1) by the linearized form $(n + 2)u\langle \phi_j^2 \rangle \phi_i$. The required expression for $\langle \phi_j^2 \rangle$ can be evaluated at $u = 0$ and $r = 0$, since r_c is also $O(u)$. For this part of the calculation, therefore, we can use the mean-field result,

$$\tilde{\phi}_{\mathbf{k}}(t) = \tilde{\phi}_{\mathbf{k}}(0)\exp(-k^2 t) + \int_0^t dt'\, \tilde{\xi}_{\mathbf{k}}(t')\exp[-k^2(t - t')]\,, \tag{10}$$

to give

$$\langle \phi_j^2 \rangle = \Delta \sum_{\mathbf{k}} \exp(-2k^2 t) + \sum_{\mathbf{k}} \frac{1}{k^2}[1 - \exp(-2k^2 t)]\,, \tag{11}$$

where we have used $\langle \tilde{\phi}_{\mathbf{k}}(0)\tilde{\phi}_{-\mathbf{k}}(0)\rangle = \Delta$, appropriate to short-range spatial correlations in the initial state.

To order the critical point is $r_c = -(n + 2)u\sum_{\mathbf{k}} k^{-2}$. The effective Langevin equation for the $k = 0$ mode, correct to $O(u)$, is therefore (suppressing the index i and the momentum subscript)

$$\partial_t \tilde{\phi} = (n + 2)u \times \sum_{\mathbf{k}} \left(\frac{1}{k^2} - \Delta\right)\exp(-2k^2 t)\tilde{\phi} + \tilde{\xi}\,. \tag{12}$$

The **k** integrals are now performed in $d = 4$. It is clear that the term involving the initial-condition correlator Δ is smaller (by one power of t) than that coming from the thermal noise, and may therefore be dropped, giving

$$\partial_t \tilde{\phi} = (n + 2)\frac{uK_4}{4t}\tilde{\phi} + \tilde{\xi}, \qquad (13)$$

where $K_4 = 1/8\pi^2$ is the usual geometrical factor. Setting u equal to its RG fixed-point value [10] $u^* = \epsilon/[(n + 8)K_4]$ gives an equation identical to the large-n equation (8), but with the replacement $\epsilon \to [(n + 2)/(n + 8)]\epsilon$. Making the same replacement in (9), we deduce immediately that the exponent θ is given by

$$\theta = \frac{1}{2} - \frac{1}{4}\left(\frac{n + 2}{n + 8}\right)\epsilon + O(\epsilon^2), \qquad (14)$$

which agrees with (9) for $n \to \infty$. For the Ising universality class ($n = 1$), (14) becomes $\theta = 1/2 - \epsilon/12 + O(\epsilon^2)$.

The final soluble limit we consider is the $d = 1$ Ising model with Glauber dynamics. For this model, the critical point is at $T = 0$, so the "critical" and "strong coupling" fixed points coincide. The persistence probability $p(t)$ for a single spin has been considered earlier [4]. Very recently, it has been shown to decay as $t^{-3/8}$, with nontrivial results for general q-state Potts models [5]. The calculation of $p(t)$ for the global magnetization $M(t)$ is much simpler. At $T = 0$ the dynamics is equivalent to a set of annihilating random walkers (the domain walls). At each time step, each random walker moves independently one step to the left or right [12]. Let the number of spins be N. Then the number of surviving walkers at time t is of order $Nt^{-1/2}$ [13,14]. The change in $M(t)$ in one time step is therefore of order $\sqrt{N}\, t^{-1/4}$, since the contributions from the walkers add incoherently. The $k = 0$ Fourier component $\tilde{\phi} = M/\sqrt{N}$ therefore satisfies the Langevin equation (up to constants) [15]

$$\partial_t \tilde{\phi} = t^{-1/4}\xi(t), \qquad (15)$$

where $\xi(t)$ is a Gaussian white noise, $\langle \xi(t)\xi(t')\rangle = C\delta(t - t')$, and C is a constant.

This can be reduced to the standard random-walk dynamics through the change of variable $t = \tau^2$. Equation (15) then reads $d\tilde{\phi}/d\tau = 2\tau^{1/2}\xi(\tau^2) \equiv \eta(\tau)$, where $\eta(\tau)$ has correlator $\langle \eta(\tau)\eta(\tau')\rangle = 4C\tau\delta(\tau^2 - \tau'^2) = 2C\delta(\tau - \tau')$, i.e., $\eta(t)$ is a Gaussian white noise. We conclude that $p(t) \propto \tau^{-1/2} = t^{-1/4}$, i.e., $\theta = 1/4$ for this model. It is remarkable, but certainly coincidental, that the $O(\epsilon)$ result gives this result exactly, on setting $\epsilon = 3$.

At this point we note a simplifying feature of all the results presented so far, namely, the underlying dynamics is a Gaussian Markov process in every case. This should be apparent from Eqs. (2), (8), (13), and (15). For such cases one can derive (see below) a scaling law relating θ to other exponents, namely,

$$\theta z = \lambda - d + 1 - \eta/2, \qquad (16)$$

where λ describes the asymptotics of the two-time correlation function of the global order parameter at T_c: $\langle \tilde{\phi}(t_1)\tilde{\phi}(t_2)\rangle = t_1^{(2-\eta)/z}F(t_2/t_1)$, with $F(x) \sim x^{(d-\lambda)/z}$ for $x \to \infty$. Using the known results $\eta = 0$, $z = 2$, $\lambda = (3d - 4)/2$ for $n = \infty$ [1], $\eta = 0$, $z = 2$, $\lambda = d - [(n + 2)/(n + 8)]\epsilon/2$ to $O(\epsilon)$ [1], and $\eta = 1$, $z = 2$, $\lambda = 1$ for the $d = 1$ Ising model [13], one can check that all of the results derived above satisfy this scaling law.

Does this scaling law hold generally? We do not think so: we believe that $\Phi(t)$ is *not* a Markov process in general (though it is Gaussian), for the following reason. Consider the autocorrelation function for the $k = 0$ mode, $\langle \tilde{\phi}(t_1)\tilde{\phi}(t_2)\rangle$. We have seen that it has the scaling form $t_1^{(2-\eta)/z}F(t_2/t_1)$, with $F(x) \sim x^{(d-\lambda)/z}$ for $x \to \infty$. Now construct the normalized autocorrelation function $a(t_1, t_2) = \langle \tilde{\phi}(t_1)\tilde{\phi}(t_2)\rangle/\langle \tilde{\phi}(t_1)^2\rangle^{1/2}\langle \tilde{\phi}(t_2)^2\rangle^{1/2}$. This has the scaling form $a(t_1, t_2) = f(t_1/t_2)$, with $f(x) \sim x^{(\lambda-d+1-\eta/2)/z}$ for $x \to \infty$. If we introduce the new time variable $T = \ln t$, this becomes $A(T_1, T_2) = g(T_1 - T_2)$, i.e., the process is a Gaussian stationary process in this time variable. Also the function $g(T)$ has the asymptotic form $g(T) \sim \exp(-\bar{\lambda}|T|)$, with $\bar{\lambda} = (\lambda - d + 1 - \eta/2)/z$.

Now if the process is Markovian, $g(T)$ necessarily has this exponential form for *all* T, not just for asymptotically large T [16]. Futhermore, the first-passage exponent θ is then equal to $\bar{\lambda}$ [9,16,17], which is the origin of the scaling law (16) for Markov processes. Note that, in the original time variables, requiring $g(T)$ to be a simple exponential is equivalent to requiring that the scaling function $f(t_1/t_2)$ be a simple power of t_1/t_2 for *all* $t_2 > t_1$, not just $t_2 \gg t_1$.

So the question of whether $\Phi(t)$ is a Markov process comes down to the question of whether the scaling function $f(t_1/t_2)$ [of the normalized autocorrelation function of $\tilde{\phi}(t)$] is a simple power law for all $t_2 \geq t_1$. The known results for $n = \infty$, $O(\epsilon)$, and the $d = 1$ Ising model satisfy this requirement. For the last of these, the reduction to a random walk makes this transparent. In the other two cases, it is consequence of the simplicity of the one-loop nature of the calculations, which give simple powers. To $O(\epsilon^2)$, however, the structure of the "watermelon" two-loop graph leads to a nontrivial dependence on t_1/t_2, which does not reduce to a simple power [18]. It follows that the putative scaling law (16) will fail at $O(\epsilon^2)$: The dynamics of the global order parameter are non-Markovian in general, and the exponent θ is an independent critical exponent. Similar conclusions follow consideration of the next term in the $1/n$ expansion.

The exponent θ was measured numerically for 2D Ising systems, using a finite-size scaling technique for square lattices of linear size $L = 24, 32, 48, 64, 96,$ and 128, with periodic boundary conditions. Starting from a random initial condition, the systems were evolved using heat-bath Monte Carlo dynamics at the bulk critical coupling $K_c = [\ln(1 + \sqrt{2})]/2$. Each system was evolved until the global magnetization first changed sign. The fraction $p(t)$ of surviving systems was then computed at each time t, over a number of runs varying from 200 000 for $L = 24$ to 91 008 for $L = 128$. Finite-size scaling suggests the scaling form $p(t) = t^{-\theta} f(t/L^z) = L^{-\theta z} \tilde{f}(t/L^z)$, where z is the dynamic exponent. We therefore plot $L^{\theta z} p(t)$ against t/L^z, and vary θ for the best data collapse. The dynamic exponent was taken to be $z = 2.172$ [19]. Data for $t < 20$ were discarded. The best collapse was obtained for $\theta z = 0.505 \pm 0.020$. Scaling plots for $\theta z = 0.485, 0.505$, and 0.525 are shown in Fig. 1.

A finite-size scaling analysis is essential here, as the data show significant curvature in the "early" time regime, even for the largest systems studied (128^2). In the scaling form $L^{\theta z} p(t) = F(t/L^z)$, the scaling function $F(x)$ must vary as $x^{-\theta}$ for small x, but the "small-x" regime in the data is not extensive enough to extract the exponent from this part of the plot alone. In the large-x regime, one expects $F(x) \sim \exp(-\mathrm{const}\,x)$, since L^z is the characteristic relaxation time of the system. This behavior is confirmed in studies of smaller systems, where longer runs are feasible. The final part of the scaling plots in Fig. 1 shows the entry into this exponential regime.

It is interesting to compare the numerical result for θz with the prediction of the "scaling law" (16). Using $\lambda = 1.585 \pm 0.006$ [19], and the exact result $\eta = 1/4$, (16) gives $\theta z = 0.460 \pm 0.006$, compared with the measured value 0.505 ± 0.020. This suggests that non-Markovian violations of the relation (16) may be small, but measurable.

In summary we have identified a new exponent θ for critical dynamics. It is the analog of the persistence exponents discussed in a number of other contexts recently, and characterizes the time dependence of the probability that the global order parameter has not changed sign up to time t after a quench to the critical point from the disordered phase. We have argued that θ is in general an independent critical exponent, not related by scaling laws to other critical exponents, although the relation (16) is exact for $n = \infty$, to first order in $\epsilon = 4 - d$, and for the $d = 1$ Ising model (for which the dynamics are Markovian). The numerical results for the $d = 2$ Ising model, however, show evidence for non-Markovian effects. The corresponding exponent for the global order parameter following a quench into the ordered phase is also of interest, and is currently under investigation by numerical simulations.

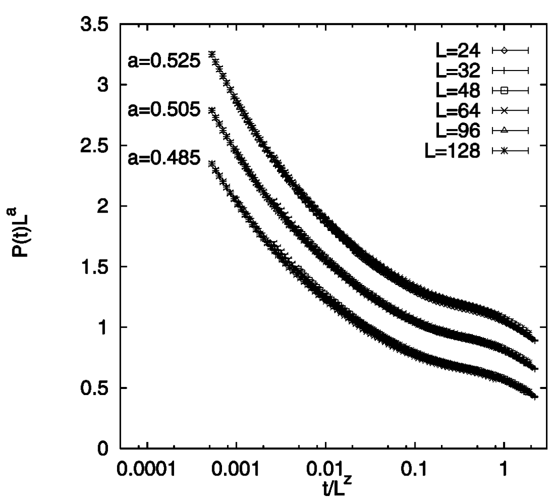

FIG. 1. Finite-size scaling plots for the "persistence" $p(t)$ (the fraction of systems whose total magnetization has not changed sign) for the $d = 2$ Ising model at T_c, with $z = 2.172$ and $a \equiv \theta z = 0.485, 0.505$, and 0.525. For clarity, the 0.505 and 0.525 data have been moved up by 0.2 and 0.4, respectively.

Note added.—Recent work by D. Stauffer has extended our Ising model simulations to $d = 3$, 4, and 5. The $d = 3$ result is in good agreement with the ϵ expansion (14), while for $d = 4$ and 5 the results are consistent with the mean-field result $\theta = 1/2$, as predicted.

A. B. and S. C.'s research was supported by EPSRC (UK), S. M.'s by NSF Grant No. DMR-92-24290. A. B. thanks SISSA (Italy), and Amos Maritan, in particular, for hospitality during the final stages of this work, and Alan McKane for discussions.

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