Scaling Laws for Fracture of Heterogeneous Materials and Rock

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Using computer simulation we show that, near the global failure point, the cumulative elastic energy released during fracturing of heterogeneous solids follows a power law with *log-periodic corrections* to the leading term. This is consistent with a recently proposed scaling law that relates the dynamics of the precursors of large earthquakes to their occurrence time, thus providing a rational basis for it in terms of rupture of the rock. It is also consistent with the scaling of acoustic emissions that precede fracture of composite materials, with the time to failure, and may thus provide a basis for predicting fracture of materials. [S0031-9007(96)01440-8]

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Two important phenomena in natural rock masses are earthquakes, and the nucleation and propagation of fractures. Most rock masses contain complex and interconnected fracture networks, the presence of which is crucial to the economics of oil recovery processes from underground reservoirs, generation of vapor from geothermal reservoirs for use in power plants, and the development of groundwater resources, as the fractures provide high permeability paths for fluid flow in natural reservoirs. Because of its obvious significance, characterization of fractured rock is an active research field [1]. Earthquakes, on the other hand, are the result of a series of complex phenomena involving the interaction between stress concentration, the structure of the fracture and fault network of rock, and local pore fluid pressures. Study of the dynamics of earthquake faults is also a very active area of research. However, a clear picture of how earthquakes develop has not emerged yet. For example, although we know that the vast majority of earthquake hypocenters are distributed on the regional fracture and fault networks, the nature of the dynamics of earthquakes and the precise relation between rupture and fracture of rock and this dynamics, which are the most important problems, have remained largely unsolved.

It has been proposed [2–4] that large earthquakes are similar to critical phenomena in that before they occur long-range correlations develop at many scales that lead to a cascade of events (earthquakes) at increasingly larger scales. The development of such long-range correlations between the events that precede a great earthquake has been documented [5]. If this picture of the development of a large earthquake is correct, then one may guess that its precursors may follow power laws which are characteristics of critical phenomena. This guess has already been exploited for predicting large earthquakes. In particular, it has been suggested [6,7] that any measure $\epsilon(t)$ of seismic release at time t, close to the time of a large earthquake t_e, should obey a power law $\epsilon(t) = A + B(t_e - t)^m$, where A and B are constants, and m is a critical exponent. Moreover, using arguments from the renormalization group theory of critical phenomena, Sornette and Sammis [7] suggested that there are significant correction-to-scaling terms to this power law, and that these corrections are *log periodic,* so that one has the following scaling law:

$$\epsilon(t) = A + B(t_e - t)^m \times \left\{1 + C \cos\left[2\pi \frac{\log(t_e - t)}{\log D} + E\right]\right\}, \quad (1)$$

where C, D, and E are also constants. Such correction terms arise if the critical exponents are complex numbers. In Eq. (1) t is made dimensionless with some suitable variable or time scale. In practice, one fits the data by Eq. (1) to estimate the various parameters and, in particular, t_e. It was shown [7] that Eq. (1) can provide accurate predictions for the time t_e at which some large earthquakes have already occurred. For example, Eq. (1) predicts that the Loma Prieta earthquake in northern California, which had a magnitude of 6.7–7.1, should occur in 1989.9 $\pm$ 0.8; the earthquake actually occurred on 17 October, 1989. Despite its potential as a predictive tool, the origin of Eq. (1) and its relation with nucleation and propagation of fractures in the rock remain unclear. If this relation can be clarified, it can lead to a much deeper understanding of the long-standing problem of the dynamics of large earthquakes. As a by-product, one may potentially have an accurate tool for predicting large earthquakes.

If large earthquakes do represent a critical phenomenon, then only may be led to the idea that the criticality is caused by failure in the Earth's crust which can be thought of as a scaling-up process in which failure at one scale is part of damage accumulation and creation of fractures at a larger scale. Hence, Eq. (1) should also be observed for any measure that characterizes nucleation and propagation of fractures in heterogeneous rock. The purpose of this Letter is to show that this is indeed the

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case, and that, near the macroscopic failure point, the cumulative elastic energy released during formation of fractures in heterogeneous rock follows an equation similar to (1). Thus, one can explain the origin of Eq. (1) based on the relation between the dynamics of large earthquakes and that of the development of rupture and fracture in heterogeneous rock, and might ultimately have an accurate means of estimating when a large earthquake may occur. However, we argue that our results are more general and may be applicable to a broad class of phenomena, ranging from earthquakes and rock fracture, the main focus of this Letter, to fracture of heterogeneous materials and the critical properties of spin systems (see below).

To establish this we employ our discrete model of fracture [8–10] for heterogeneous media. Consider an $L \times L$ triangular network with a periodic boundary condition in one direction, each site of which is characterized by the displacement vector $\mathbf{u}_i = (u_{ix}, u_{iy})$, with nearest-neighbor sites connected by springs, where each spring represents a portion of rock at a small scale. We consider here the case of brittle fracture for which a linear approximation is valid up to a threshold (defined below). The displacement $\mathbf{u}_i$ is computed by minimizing the elastic energy E with respect to $\mathbf{u}_i$, where

$$E = \frac{\alpha}{2} \sum_{\langle ij \rangle} [(\mathbf{u}_i - \mathbf{u}_j) \cdot \mathbf{R}_{ij}]^2 g_{ij} + \frac{\beta}{2} \sum_{\langle jik \rangle} (\delta\theta_{jik})^2 g_{ij} g_{ik} \,. \qquad (2)$$

Here α and β are the central and bond-bending or angle-changing force constants, respectively, $\mathbf{R}_{ij}$ is a unit vector from site i to site j, g_{ij} is the elastic constant of the spring between i and j, and $\langle jik \rangle$ indicates that the sum is over all triplets in which the bonds j-i and i-k form an angle whose vertex is at i. We introduce a threshold value l_c for the length of a spring, which is selected from a probability distribution. Two types of threshold distributions were used. One was a power law, $f(l_c) = (1 - \gamma) l_c^{-\gamma}$, with $0 < \gamma < 1$. Thus, for $\gamma \simeq 0$ one has a narrow distribution of the thresholds, whereas for $\gamma \simeq 1$ one has a very broad and heterogeneous distribution. However, this distribution is completely random and provides no correlations between various regions of the system. It has been shown [1,11,12] that in natural rock at large length scales there are long-range *anticorrelations*, in the sense that a high value of a rock property (e.g., its porosity or the elastic moduli) is followed by a low value, and vice versa. To generate such correlations we used a fractional Brownian motion (fBm) $B_H(\mathbf{r})$ that has been shown [1,11] to provide adequate representations of such correlations in rock, although any other stochastic process that can generate such correlations can be used. Briefly, the fBm is a stochastic process such that $\langle [B_H(\mathbf{r}) - B_H(\mathbf{r}_0)]^2 \rangle \sim |\mathbf{r} - \mathbf{r}_0|^{2H}$, where $\mathbf{r}$ and $\mathbf{r}_0$ are two arbitrary points in space, and H is the Hurst exponent. It has been shown [13] that fBm generates correlations whose extent is infinite (i.e., as large as the linear size of the system). For $1/2 < H < 1$ a fBm generates long-range positive correlations, whereas for $0 < H < 1/2$ one has negative or anticorrelations. In our simulations we used $H = 0.1$ and 0.9, where the smaller value is about what has been found for rock [12].

We initiate the failure process by applying a fixed external strain to the network in a given direction (in shear or tension), calculating $\mathbf{u}_i$'s, and breaking all the springs whose lengths have exceeded their critical threshold l_c, where each broken spring represents a microcrack. We then increase the external strain gradually and recalculate $\mathbf{u}_i$'s for the new configuration of the network, select the next springs to break, and so on. At each stage we also calculate the cumulative elastic energy ΔE that is released by fracturing of the system, i.e., the total energy E_0 of the system when no fracture has been created minus the elastic energy of the system in its current state. The simulation continues until a sample-spanning fracture network is formed. We used an 80×80 network with $\beta/\alpha = 0.05$, which is a typical value for heterogeneous solids, and varied γ in order to study its effect on the behavior of the system. We also made several realizations of the system, and studied its behavior in individual realizations, as well as its average over all the realizations (see below).

Let us discuss first the results with the random distribution of the thresholds. If $\gamma \simeq 0$, then the system is more or less homogeneous. Under this condition, once a crack nucleates in the rock, stress enhancement at its tip is larger than at any other point of the medium, and therefore the next microcrack almost surely develops at its tip. Hence, in such a system the microcracks are all connected and clustered together with almost no dead-end branches, and one does not see large fluctuations in, e.g., the released elastic energy which is typical of the precursors to large earthquakes. On the other hand, if $\gamma \simeq 1$, fracture of the system is catastrophic in the sense that, with a very small increase in the external strain a very large number of fractures are created and the system reaches its final state very quickly, and thus it is very difficult to detect the fluctuations, unless one uses very large systems. It is only in the intermediate region where we see, without difficulty, clear fluctuations in the cumulative elastic energy released during fracturing of the system. Figure 1 shows the results for a single realization and $\gamma = 0.5$, where we present both our simulation results and their fit to Eq. (1), with the time t replaced by the external strain S which plays the same role as t in our model. The cumulative energy has been normalized by E_0, while S has been normalized by its value S_c just before a sample-spanning fracture network has been formed. The fits were obtained using a Levenberg-Macquardt algorithm. As can be seen, Eq. (1) provides an accurate fit to the data with $m \simeq 0.13$, whereas a simple power law without the log-periodic corrections does not. Note that the last data point, which represents the global failure point, is not on the fitted curve,

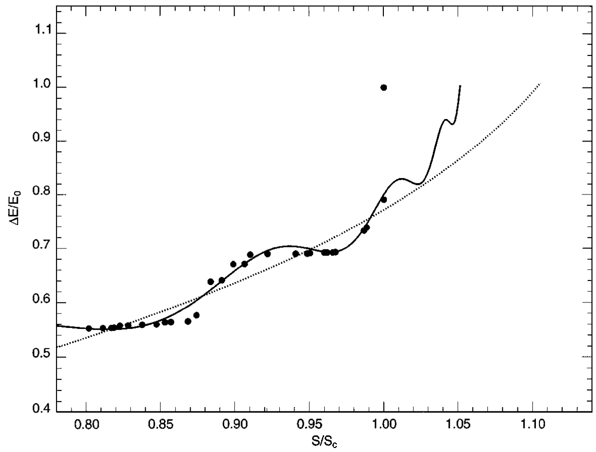

FIG. 1. Variations of the cumulative elastic energy ΔE released during fracturing (normalized by the energy E_0 of the unfractured system) with the external strain S, normalized by its value S_c just before the global failure, for a single realization of a random system. The dashed curve shows the fit of the data (circles) by a simple power law, whereas the solid curve shows the fit with log-periodic corrections with the power law.

but to the left of it (failure occurs early). However, this is a finite-size effect which will disappear if the size of the network becomes very large.

An important question is whether such a behavior will disappear if we average the behavior of the system over many realizations. Figure 2 presents the same type of data as those in Fig. 1, except that now the results represent the average of several realizations of the system. As can be seen, the oscillatory fluctuations do not die out when the behavior of the system is averaged over many realizations. For this case we find $m \simeq 0.1$ which, as expected, is close to what we find for a single realization. Since the fracturing process is nonlinear, we do not expect, on theoretical grounds [H. Saleur (private communications)], that the oscillations disappear after the averaging.

We now discuss the results with a correlated distribution of the thresholds, which we believe is more relevant to heterogeneous rock at large length scales. If $H > 1/2$, then the springs with low thresholds are clustered together. Since the first microcrack appears in a weak region of the system, the positive correlations cause the crack to grow in the same region, and as a result the behavior of the system is essentially similar to the random case with $\gamma \simeq 0$, and indeed this is what we find with $H = 0.9$. However, with $H < 1/2$, which is the range relevant to heterogeneous rock [12], we have anticorrelations, so that high and low values of the thresholds are next to each other. As a result, crack growth continues in a weak region until a much stronger region is encountered in its neighborhood. When this happens, crack growth stops and another crack nucleates in another weak region. The growth of the new crack also stops when it encounters another strong region, and so on. This then gives rise to large fluctuations in the cumulative elastic energy released during fracturing, and indeed Fig. 3 which presents the results with $H = 0.1$ confirms this. As in Figs. 1 and 2, Eq. (1) provides an accurate fit to our data with $m \simeq 0.4$, close to what Sornette and Sammis [7] found in their analysis of Loma Prieta earthquake data, $m \simeq 0.34 \pm 0.08$. We do not expect the value of m for this case to be the same as that of the random distribution, since long-range correlations change the value of a critical exponent from its value for a random system.

The existence of such corrections to the scaling of the released elastic energy may be explained as follows. The rock first develops isolated microcracks which are nucleated in the weak regions. As the applied strain increases, more weak regions develop cracks, while the stronger regions remain relatively intact. After some time the microcracks join and a fractal fracture pattern emerges [8–10] which explains the existence of the power law in Eq. (1). At the same time, because the weaker regions have al-

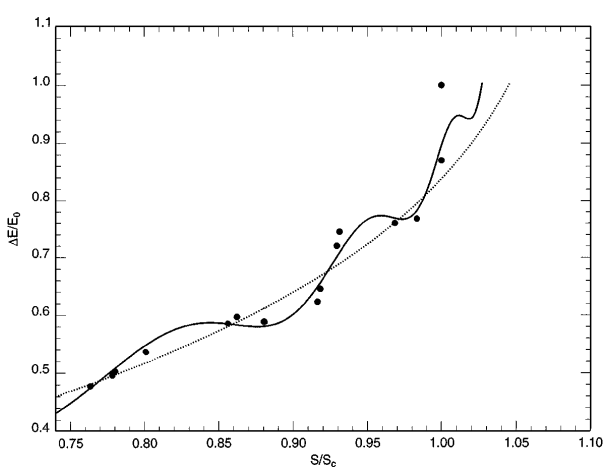

FIG. 2. The same as in Fig. 1, but averaged over several realizations of the system.

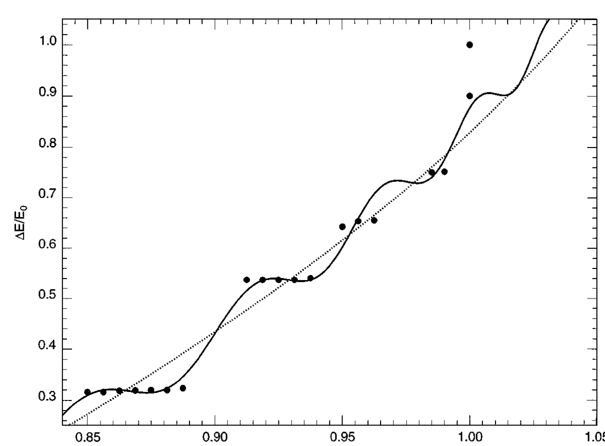

FIG. 3. The same as in Fig. 1, but for a correlated system.

ready failed, for a range of the applied strain no major crack is formed, and the released elastic energy varies little until a value of the strain S is reached at which a number of cracks can form, and therefore the released energy increases sharply. This is repeated in cycles that are, similar to the earthquake data [6,7], of increasingly smaller sizes in the applied strain (time interval in the earthquake data), but larger in the released energy, since as more fractures are formed one nears the critical region in which more and more cracks are created with an increasingly smaller change in the applied strain, until the sample-spanning fracture network is formed and global failure occurs. Thus, we may view seismicity in a region as a sequence of fracturing cycles, where each cycle represents a progressive cooperative stress buildup and crack nucleation that culminate in some sort of a critical point—the formation of the sample-spanning fracture network—which is characterized by global failure in the form of a large earthquake. Saleur, Sammis, and Sornette [14] have suggested that the existence of *discrete scale* invariance is essential for having log-periodic corrections. Our results indicate that the interplay between the heterogeneities of rock and the stress field generates *dynamically* such a discrete scale invariance, and does not have to be present in the rock structure itself.

We now discuss the evidence that our results are more general, and may be applicable to a broader class of phenomena than fracture of rock and earthquakes. In a recent fracture experiment with carbon fiber-reinforced resin [15] the rate of acoustic emissions, which preceded the macroscopic fracture, was found to follow an equation similar to (1). Thus, our results may also have a practical implication for predicting fracture of heterogeneous materials, an outstanding unsolved problem: one fits the rate of acoustic emissions, a measurable quantity, to an equation similar to (1) to predict the time t_e at which the material will fail. In Ref. [4] a time-dependent *hierarchical* model of earthquakes was considered, the solution of which indicated the existence of log-periodic corrections. However, in our model there is no built-in hierarchical structure. Finally, log-periodic corrections have been shown to be important in the critical properties of spin systems, if they are defined on a hierarchical lattice [16]. Our results indicate again that log-periodic corrections are a much more general phenomenon than previously thought.

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