# Probabilistic Fragmentation and Effective Power Law

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A probabilistic fragmentation model is introduced and analyzed. We show that, under very general conditions, an effective power law for the mass distribution arises with a realistic exponent. The exponent has a universal limit, but, in practice, the effective exponent depends on the detailed breaking mechanism and the initial conditions. This dependence is in fair agreement with experimental results. [S0031-9007(96)01534-7]

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One of the best known physical processes in nature is fragmentation. From our daily experience we know that a material bulk under stress or shock will break into smaller pieces. Experiments show generally that the number of fragments with a linear size larger than $r$ behaves such as

$$N(r) \sim r^{-D}. \tag{1}$$

The exponent $D$, usually called fractal dimension, is found to lie in the range $D \approx 2-3$ for fragmentation of three-dimensional objects [1]. A theoretical understanding of the statistical origin of Eq. (1) is currently being pursued by many authors. The simplest models [2] predict a log-normal distribution, incompatible with observation. More refined models, with various assumptions about breaking mechanism [1,3], yield a power law behavior for $N(r)$. However, it appears that theories producing a single universal exponent may not account for the experimental range of $D$. Recently, quite a large amount of numerical simulations [4] with rather realistic physical parameters such as stress, shear, and neighborhood were able to show qualitatively correct power laws.

In this Letter we develop an analytical model of fragmentation, *without* using a specific breaking mechanism. We shall see that under very general, simple conditions an effective power law will result. A general theory is called for, since it is difficult to observe what is really happening *during* fragmentation. We, nevertheless, assume the folklore that fragmentation happens in a hierarchical order—i.e., a large piece first breaks into $n$ smaller ones, and these fragments may then break further. For simplicity we assume $n = 2$ for all levels. Our numerical simulations show that other finite $n$ does not affect our conclusions.

At level $k$ of the hierarchy, we consider one object of volume $V$ and energy $E$, the only variables retained in our model. $E$ is the total energy, including kinetic energy, elastic energy, etc. The only property of the energy we retain is that it is conserved.

Our model describes fragmentation of free flying objects, thus excluding the other interesting domain where the fractured matter is confined [5]. Even for free objects, the neglect of dissipation is an approximation. This is reasonable for large masses because usually dissipation is an effect proportional to the area of the fracture produced. The main effect of dissipation is to set a scale where the fragmentation cascade stops. Indeed, all the input energy will ultimately be dissipated. In an analogy with turbulence, we focus here on the "inertial range" well above the dissipation scale.

We assume further that it is the energy density $E/V$ that decides whether an object breaks: If $E/V$ exceeds a threshold, set to 1 for all $k$, the object breaks further, otherwise not. If $E/V > 1$, the object breaks into two pieces of energy, $\varepsilon$ and $E - \varepsilon$, and volume, $v$ and $V - v$, respectively. The two resulting fragments will or will not break in their turn according to their energy-volume ratios, i.e., if

$$x_1 = \frac{\varepsilon}{v} > 1, \qquad x_2 = \frac{E - \varepsilon}{V - v} > 1. \tag{2}$$

The above process is repeated for an arbitrary number $k$ of levels. Note that $k$ is *not* necessarily proportional to time. The above variables are all for level $k$, and we have suppressed the subindex $k$ for clarity. At level $k = 0$, $E_0$ and $V_0$ are given by the initial energy and volume of the system ($E_0/V_0 > 1$).

Let $q(\varepsilon, v \,|\, E)d\varepsilon dv$ be the probability that the energy and volume of an element are between $\varepsilon$ and $\varepsilon + d\varepsilon$, and $v$ and $v + dv$, given that it results from the fragmentation of an object of unit volume $V = 1$ and energy $E > 1$ (for our purposes we may consider the volume $V$ to be unity at any level, since only the ratio $E/V$ matters). This distribution accounts for all the information of a detailed breaking mechanism. We shall assume that $q(\varepsilon, v \,|\, E) = \frac{1}{E}\varphi(v)$. This implies a uniform distribution in energy, but arbitrary distribution in volume [for symmetry we require $\varphi(v) = \varphi(1 - v)$]. This certainly is not the most general case, but it still includes a large class of models.

The fragments with energy density $x > 1$ are called "unstable," those with $x < 1$ are called "stable." The nature of the cascade process is best illustrated by studying the distribution $p_k(x)$ of energy density $x$ of the *unstable* elements at the level $k$. This can be computed from the initial distribution $p_0(x)$ once a recurrence relation between $p_k(x)$ and $p_{k+1}(x)$ is found. In order to

  

derive this relation, let us consider a unit volume object with energy $E > 1$ breaking in two smaller elements. The Jacobian of the transformation $(\varepsilon, v) \to (x_1, x_2)$ is readily found using (2), and we find the joint distribution of $x_1$ and $x_2$ from the above $q(\varepsilon, v \mid E)$,

$$p(x_1, x_2) = \frac{1}{E}\varphi\left(\frac{E - x_2}{x_1 - x_2}\right)\left[\frac{(x_1 - E)(x_2 - E)}{(x_1 - x_2)^3}\right]. \quad (3)$$

Denoting $x_+ = \max(x_1, x_2)$ and $x_- = \min(x_1, x_2)$, from (2) we have $x_- \le E \le x_+$. Integrating out one of the two variables, we obtain the distributions of $x_\pm$, respectively,

$$p_+(x|E) = \frac{2\theta(x - E)}{E}\int_0^{E/x} v\varphi(v)dv\,,$$

$$p_-(x|E) = \frac{\theta(E - x)}{E}\,, \quad (4)$$

where $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ otherwise. The $(+)$ element always breaks further since $x_+ \ge E > 1$, while the $(-)$ element does not break if $x_- < 1$; this occurs with the probability

$$C(E) = \int_0^1 p_-(x|E)dx = \frac{1}{E}\,. \quad (5)$$

Let us define the distribution of the energy density $p(x|E)$ of one *unstable* $(x > 1)$ element between the two fragments. The pair $x_-$, $x_+$ falls into two cases. (i) If $x_- < 1$, which occurs with probability $C(E)$ given above, the only possibility is that $x = x_+$. In this case, the distribution $p(x|E)$ is just that of $x_+$, i.e., $p_+(x|E)$. (ii) If $x_- > 1$, which occurs with probability $1 - C(E)$, $x$ can be either $x_+$ or $x_-$. In this case, occurring with the probability $1 - C(E)$, $p(x|$ is the average of the distributions $p_\pm(x|E)$, conditional to $x > 1$, with weight $1/2$ each. Taking into account the two cases, we find

$$p(x|E) = \theta(x - E)\frac{E + 1}{E^2} \times \int_0^{E/x} v\varphi(v)dv + \theta(E - x)\frac{1}{2E}\,.$$

One can verify that $p(x|E)$ is normalized in $[1, \infty)$. Using $p(x|E)$, we finally construct the iteration relation,

$$p_{k+1}(x) = \int_1^\infty dE\,p(x|E)p_k(E)\,. \quad (6)$$

As $k \to \infty$, the distribution $p_k(x)$ is expected to converge to a limit which describes the asymptotic behavior

$$p_\infty(x) = \int_1^\infty p(x|E)p_\infty(E)dE\,. \quad (7)$$

The solution of this equation will depend on the specific choice of $\varphi(v)$. For the special case $\varphi(v) = 1$, i.e., the uniform distribution which is interesting *per se*, Eq. (7) can be solved to give $p_\infty(x) = \frac{A}{x}(1 - \frac{1}{2x})\exp(-\frac{1}{2x})$. This distribution is not normalizable because $p_\infty(x) \sim A/x$ for large $x$. For arbitrary $\varphi(v)$, a detailed study of Eq. (7) for $x \to \infty$ reveals that this asymptotic behavior holds as well. Indeed, setting $p_\infty(x) \sim Ax^{-\gamma}$ in Eq. (7) and carrying out an asymptotic analysis, one finds that the exponent $\gamma$ must satisfy the equation

$$\gamma = \int_0^1 v^{1-\gamma}\varphi(v)dv\,. \quad (8)$$

This equation always has a solution for $\gamma = 1$. For convex $\varphi$, which implies easier breaking at the middle than at the edges, a further subleading solution $\gamma > 1$ exists. For concave $\varphi$, which represents, e.g., breakoffs from abrasion, the second solution is $\gamma < 1$, but it is never reached under iteration. Thus $\gamma = 1$ is the physical solution for any $\varphi$.

Denote by $C_k$ the probability that one of the fragments (the other is unstable by definition) becomes *stable* at level $k$:

$$C_k = \int_1^\infty C(E)p_k(E)dE\,. \quad (9)$$

The important consequence of $\gamma = 1$ is that $C_k \to 0$ as $k \to \infty$, regardless of the distribution $\varphi(v)$. Indeed, $C_k = 2p_{k+1}(1)$ and $\gamma = 1$ implies $p_k(1) \to 0$ as $k \to \infty$.

Unfortunately, we cannot obtain $C_k$ in closed form even for the uniform distribution $\varphi = 1$. The numerical procedure is straightforward: start with a distribution $p_0(E) = \delta(E - E_0)$ at level $k = 0$ $(E_0 > 1)$, iterate Eq. (6) to the desired level $k$, then find $C_k$ using (9). The numerical results for a particular $\varphi$ and various initial energies are plotted in Fig. 1. The function $C_k$ is not universal. It depends on $E_0$ as well as on the function $\varphi$. In general, for a given $\varphi$, $C_k(E_0)$ decreases with $E_0$ for fixed $k$, and vanishes slowly when $k \to \infty$ (see Fig. 1).

From the above preparations, we can establish the desired scaling laws. In Eq. (1) only the *stable* fragments left

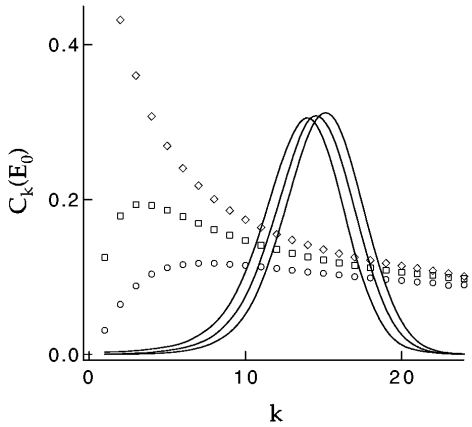


FIG. 1. $C_k$ as a function of $k$ for $V_0 = 1$ and $E_0 = 2$ $(\diamond)$, $E_0 = 8$ $(\square)$, and $E_0 = 32$ $(\circ)$. The data refer to the tent distribution $\varphi(v) = 4v$ for $v < 1/2$ and $\varphi(1 - v) = \varphi(v)$. The lines refers to the contribution $\int_{V_{\min}}^1 W_k(V)dV$ of the level $k$ to the statistics of $W(V)$ for $V_{\min} < V < 1$ with $V_{\min} = 3 \times 10^{-5}$. These lines refer, from left to right, to the same values of $E_0 = 2, 8$, and 32 used for $C_k$.

in the cascade count. $N(r)$ is trivially related to the distribution $W(V)$ of volumes $V = r^d$: $N(r) = \int_V^\infty W(V')dV'$, where $W(V)dV$ is the number of stable fragments with volume between $V$ and $V + dV$. We thus have the relation from (1):

$$W(V) \propto V^{-\alpha-1}, \qquad D = d\alpha . \tag{10}$$

$W(V)$ receives contributions $W_k(V)$ from all fragmentation levels and it can be written

$$W(V) = \sum_{k=1}^{\infty} W_k(V) \simeq \sum_{k=1}^{\infty} C_k N_k w_k(V) . \tag{11}$$

Here $w_k(V)$ is the volume distribution of all the fragments (stable and unstable) produced by the $k$th step of fragmentation, $N_k$ is the number of *unstable* fragments at level $k$. $N_k$ unstable objects produce $2N_k$ fragments, of which $(2 - C_k)N_k$, which is $N_{k+1}$, are unstable again. Thus $C_k N_k$ *stable* fragments are produced at level $k$. Equation (11) assumes that stable fragments are produced at level $k$ with a probability $C_k$ which is independent of their volume $V$. This is clearly an approximation because in reality $C_k = C_k(V)$ [6]. Neglecting this dependence allows us to keep the discussion at an elementary level. Moreover, for distributions $\varphi(v)$ which are sharply peaked around $v = 1/2$, this approximation is reliable. Indeed, for $\varphi(v) = \delta(v - 1/2)$ the dependence on the volume clearly drops (all the fragments at level $k$ have the same volume). For general distributions this approximation yields an upper bound to the true exponent $\alpha$ [6]. We shall see below that even for broad distributions, such as the uniform one, we recover qualitatively correct results. As we shall see the fractal dimension is determined by the exponential behavior of $N_k$ with $k$, which depends on $C_k$, and the explicit factor $C_k$ in Eq. (11) does not play any role. This observation supports the present approximation.

For the uniform case, $\varphi(v) = 1$, it is easy to find [7] that $w_k(V) = (\ln V)^{k-1}/(k - 1)!$. Assuming $C_k N_k \propto (2 - C^*)^k$, one can easily sum Eq. (11) with the result $W(V) \propto V^{-\alpha-1}$ with $\alpha = 1 - C^*$. Let us generalize this analysis for an arbitrary distribution $\varphi(v)$. Let us evaluate the $m$th moment of $V$ using (11) and (10). Note that some moments can be divergent and there is a smallest $m^*$ for this to happen. Multiply both sides of (11) by $V^m$ and integrate over $V$. On the right-hand side, one finds $\langle V^m \rangle_k$, where the average is done using the distribution $w_k(V)$. Since $V$, within our approximation, is the product of $k$ independent variables, each distributed by $\varphi$, we have $\langle V^m \rangle_k = [\int v^m \varphi(v) dv]^k$. Multiplying $\langle V^m \rangle_k$ by $C_k N_k$ and summing over $k$, we see (e.g., by the ratio method) that the sum first diverges when $(2 - C^*) \int v^{m^*} \varphi(v) dv = 1$, where $C^*$ is defined by $N_{k+1} = (2 - C^*)N_k$. This divergence must be matched to the one occurring on the left-hand side of the equation, which is proportional to $\int dV V^{m-\alpha-1}$. The latter occurs for $m^* = \alpha$, which therefore gives the relation

$$\int_0^1 v^\alpha \varphi(v) dv = \frac{1}{2 - C^*} , \tag{12}$$

which implicitly yields the exponent $\alpha$ within the present approximation. For $\varphi = 1$, Eq. (12) reduces to the result we found previously. Most importantly, for $C^* = 0$, and only for this value, one finds $\alpha = 1$, regardless of $\varphi$.

In order to derive the value of $C^*$, which determines the scaling exponent, we observe that in real life the observation of $W(V)$ is limited to a finite window $V_{\min} < V < V_0$, which may cover several decades. $C_k$ is a slowly varying function of $k$, whereas the typical $V$ decreases exponentially with $k$. Thus, over an exponentially large range of $V$, $C_k$ can be regarded as constant, and an *effective* power law can be established. More precisely, $W(V)$ in this window is dominated by contributions around a certain $k^*$ (see Fig. 1) thus giving $C^* = C_{k^*}$. Further fragmentation for $k > k^*$ seldom adds stable fragments larger than $V_{\min}$. Pursuing this argument further, one can find that, to leading order, $k^* \simeq c \log(V_0/V_{\min})$, where $c$ is a coefficient of order unity ($c = 1 + \alpha$ for the uniform case within our approximation). $k^*$ appears to depend weakly on the initial energy $E_0$, as shown in Fig. 1. Therefore, strictly speaking, there is no *true* power law, except at the ideal limit $k^* \to \infty$, where one can take $C^* = 0$ so that $D = d$.

How does the exponent $\alpha$ vary in practical situations? We notice that often, in experiments, $V_{\min}$ is set by a dissipation scale, below which an object cannot break anymore. In other words, at this scale, the energy lost in the rupture of an object becomes non-negligible with respect to its energy. Thus we conclude that $\alpha$ depends on $V_0$ through the dependence of $C^* = C_{k^*}$ on $k^* \sim \log(V_0/V_{\min})$. This suggests that for $V_0/V_{\min} \to \infty$ it is possible to recover the ideal limit $D = d$. Moreover, $C^*$ depends also on $E_0$. A larger $E_0$ results in a smaller $C^*$, since $C_k$ decreases with $E_0$ (see Fig. 1). Thus the effective $\alpha(E_0)$ increases with $E_0$ until it reaches the universal value of 1.

Figure 2 shows that, in spite of the approximation used to derive Eq. (12), this scenario is confirmed by numerical simulations. It also has many features that have been observed in real experiments. In the fragmentation of two-dimensional glass plates [8], the fractal dimension was nicely extrapolated to $D = 2$ for infinite input energy. The same agreement can be found comparing our results with three-dimensional data [1]. We note, in particular, that $D$ for stony meteorites and asteroids, debris resulting from extremely long fragmentation, are in excellent agreement with $D = 3$. For these systems we expect $V_0/V_{\min}$ and $E_0/V_0$ to be extremely large. On the other hand, in projectile fragmentation, the input energy is the kinetic energy of the projectile. The volume to be considered is the total volume of the system, which is essentially that of the target object. Therefore, even for extremely high kinetic energies, the input energy density $E_0/V_0$ which would enter our calculation can be relatively small. Indeed, experiments of projectile fragmentation yield exponents $D \approx 2.5$, below the universal value

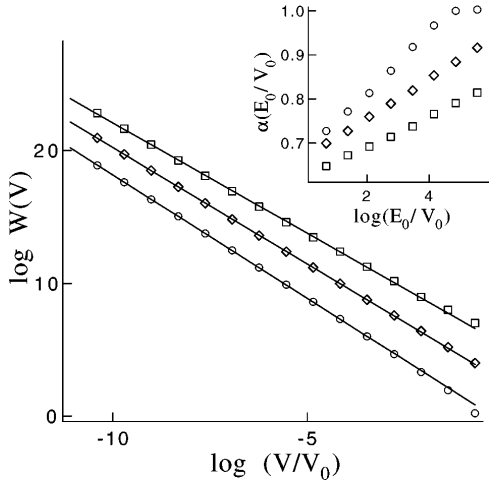


FIG. 2. Fit of the mass size distribution for $\varphi(v) = 1$ and $E_0/V_0 = 4$ (□), for the tent distribution and $E_0/V_0 = 8$ (◇), and for $\varphi(v) = \delta(v - 1/2)$ and $E_0 = 16$ (∘). Inset: exponent $\alpha$, obtained from the fit, versus $E_0/V_0$ for the uniform (□), the tent (◇), and the delta (∘) distributions.

$D = 3$. Our analysis yields $d$ as an upper bound for $D$ ($C^* \geq 0$ implies $\alpha \leq 1$). It is reassuring that, apart from the data concerning ash and pumice, materials which probably need a separate treatment, all the data in Ref. [1] have $D \leq d = 3$. Comparing the exponents obtained for different materials, in terms of our model, translates into computing the exponent for different distributions $\varphi(v)$.

Problems arise when considering the $d = 1$ experiments described, e.g., in Ref. [9]. There it was found that long thin glass rod fragmentation produces a size distribution with an exponent $D \approx 1.5$. This is clearly inconsistent with our analysis, since it would need a $C^* < 0$. This failure, we believe, lies in the assumption that the breaking of a large object is determined by its global energy density. Without loss of generality we know that energy correlation inside a volume propagates via nearest neighbor interaction; this leads to a Laplace equation. For correlation induced by a Laplace equation, we know that for $d > 2$ the correlation is very strong, and $d = 2$ is the marginal case. This strong correlation allows to describe the fragmentation of one object as an event which produces two objects and which depends on a single variable, its energy density. For $d = 1$, this is not true. The energy is very loosely correlated along a line. We suspect that, because of the weak correlation, simultaneous breaking will happen in many uncorrelated regions of a large $d = 1$ object, making our scenario invalid. For smaller and smaller rod lengths, the energy correlation becomes stronger and stronger. We therefore expect that below a certain length threshold our scenario can be applied. Remarkably, in experiments of fragmentation of long glass rods [9], a crossover occurs, and for intermediate sizes the mass distribution is described by an exponent $D \approx 0.6 < 1$.

In this work we have analyzed a simple fragmentation model. We show that, under very general conditions, an effective power law arises. The exponent is not universal but depends on the detailed mechanism and the initial conditions. There is an ideal universal limit, independent of any of our choices, which can be approached for higher input energies.

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[6] This results from observing that, in general, $C_k(V)$ should be an increasing function of $V$: Larger fragments, at the same level $k$, have a larger probability of being stable than smaller ones because their energy is assigned independently.
[7] Indeed, the volume of an element at level $k$ is $V_k = \prod_{j=1}^{k} v_j$, where $v_j$ are uniform variables. Then $-\ln V_k$ is a sum of $k$ exponential random variables, and it has a gamma distribution with index $k - 1$.
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