# Nonlinear Saturation of an Electrostatic Wave: Mobile Ions Modify Trapping Scaling

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The amplitude equation for an unstable electrostatic wave in a multispecies Vlasov plasma has been derived. The dynamics of the mode amplitude $\rho(t)$ is studied using an expansion in $\rho$; in particular, in the limit $\gamma \rightarrow 0^+$, the singularities in the expansion coefficients are analyzed to predict the asymptotic dependence of the electric field on the linear growth rate $\gamma$. Generically $|E_k| \sim \gamma^{5/2}$, as $\gamma \rightarrow 0^+$, but in the limit of infinite ion mass or for instabilities in reflection-symmetric systems due to real eigenvalues the more familiar trapping scaling $|E_k| \sim \gamma^2$ is predicted. [S0031-9007(96)01475-5]

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The evolution of an unstable electrostatic mode is a fundamental problem in collisionless plasma theory, and is perhaps the simplest nonlinear problem requiring a self-consistent treatment of the resonant interaction between waves and particles. When resonant particles interact with a large amplitude wave, then much of the behavior can be understood by analyzing the particle motion as if the wave amplitude were constant; this approximation linearizes the problem [1]. Alternatively, if the wave amplitude is sufficiently small, then the initial instability can be predicted treated by neglecting the effect of the wave on the particles; this leads to conventional linear Vlasov theory. However, to describe the dynamics of the unstable mode which develops from a small initial amplitude into a final nonlinear state requires an analysis of the self-consistent and nonlinear interaction between the wave and the resonant particles.

From a dynamical systems viewpoint even the simplest examples of instabilities in a Vlasov plasma have many unusual features related to the Hamiltonian character of the dynamics and the central role played by the neutrally stable continuous spectrum (van Kampen continuum) in the appearance of the unstable modes [2]. These novelties are not present solely in Vlasov theory; entirely analogous features arise in models of unstable inviscid shear flows, in stability calculations for certain classes of solitons, and in theories of large systems of coupled oscillators [3–9]. In the better understood setting of dissipative systems, the nonlinear evolution of the mode amplitudes can be described using an expansion in the amplitude of the unstable modes. Since the growth rates are very small near onset, nonlinear effects often act to saturate the instability before the amplitudes grow appreciably; for this reason such expansions have proved a powerful tool for studying the nonlinear states emerging from the bifurcation [10,11].

It has long been hoped that similar methods could be applied to the Vlasov equation despite the absence of dissipation. However, for many years, efforts to construct such expansions, even for the case of a single unstable mode, have been plagued by the fact that the nonlinear terms involved divergent integrals due to resonant denominators. Thus the calculations appeared to break down precisely in the regime where the amplitudes of the unstable waves were extremely small. In addition, efforts to regularize the expansion coefficients inevitably led to theories that predicted scaling behavior for the saturated amplitudes that contradicted numerical results [12,13]. More precisely, these theories predicted that the electric field of the saturated mode would satisfy $E \sim \sqrt{\gamma}$ as $\gamma \rightarrow 0^+$, whereas numerical simulations find the exponential growth of the mode halted at an amplitude characterized by the "trapping scaling" $E \sim \gamma^2$ [14,15].

Recently, we have made progress on this problem for the Vlasov equation; both in the approach to constructing the expansions and in the way the singular limit $\gamma \rightarrow 0^+$ is treated and interpreted [16]. An amplitude equation for an unstable mode in a one-dimensional collisionless plasma was derived for the dynamics on the two-dimensional unstable manifold of the equilibrium $F_0$. The essential difference from previous work lies in the choice of unperturbed state. Earlier theories assumed an equilibrium with a neutrally stable mode and obtained ill-defined expansion coefficients [12]. This can be avoided by taking the weakly unstable equilibrium as the unperturbed state; a choice that naturally leads one to work with the unstable manifold. The mode eigenvalue $\lambda = \gamma - i\omega$ can be complex (beam-plasma) or real (two-stream); in either case the equations for the amplitude $A(t) = \rho(t)\, e^{-i\theta(t)}$,

$$\dot{\rho} = \gamma\rho + a_1\rho^3 + a_2\rho^5 + \mathcal{O}(\rho^7), \quad (1)$$

$$\dot{\theta} = \omega + a_1'\rho^2 + a_2'\rho^4 + \mathcal{O}(\rho^6), \quad (2)$$

lead to a one-dimensional problem for $\rho$ because the spatial homogeneity of the equilibrium decouples the phase dynamics. As $\gamma \rightarrow 0^+$, the expansion coefficients diverge,

$$a_j,\, a_j' \sim \frac{1}{\gamma^{4j-1}}, \quad (3)$$

but these divergences can be removed to *all* orders in the expansion by rescaling the mode amplitude: $\rho(t) \equiv \gamma^2\, r(\gamma t)$. In this way one obtains asymptotic equations for $r(\tau)$ that are well behaved as $\gamma \rightarrow 0^+$, and moreover through Poisson's equation this rescaling implies that the electric field exhibits the trapping scaling. These initial results were obtained for a plasma of mobile electrons with infinitely massive ions providing a fixed neutralizing background; consequently they contain no information regarding unstable ion-acoustic modes, for example.

In order to study the effects of ion dynamics, we have generalized the analysis to treat a single unstable electrostatic mode in a *multispecies* one-dimensional plasma. This changes the problem in a qualitative way: now as $\gamma \to 0^+$, the single particle dynamics and collective motions of the ions occur on a fast time scale relative to $1/\gamma$. We calculate explicit expressions for the leading nonlinear coefficients $a_1$ and $a_1'$; in addition, we have determined the dominant singularities in the amplitude expansion to all orders. The results show a qualitatively different singularity structure from the limiting model (3) with fixed ions, and provide new predictions for the scaling of nonlinearly saturated modes.

The theory is described for the simplest example of a neutral plasma with two species ($s = e, i$) which we refer to as "electrons" and "ions," although the results apply equally to collisionless electron-positron plasmas. Let $n_s = N_s/L$ denote the average species density in a one-dimensional plasma of length $L$, and $eq_s$ denotes the charge per particle of species $s$. In convenient dimensionless variables, the Vlasov-Poisson system becomes

$$\partial_t F^{(s)} + v\partial_x F^{(s)} + \kappa^{(s)} E\, \partial_v F^{(s)} = 0,$$

$$\partial_x E = \sum_s \int_{-\infty}^{\infty} dv\, F^{(s)}, \tag{4}$$

where $\kappa^{(s)} \equiv (q_s m_e/m_s)$. We assume periodic boundary conditions and adopt the normalization $\int dx \int dv\, F^{(s)} = q_s n_s L/n_e$; note that $F^{(s)}$ is negative for electrons and positive for ions. Given a spatially homogeneous equilibrium $F_0^{(s)}(v)$, this system determines an evolution equation for $f^{(s)}(x,v,t) \equiv F^{(s)}(x,v,t) - F_0^{(s)}(v)$: $\partial_t f^{(s)} = \mathcal{L} f^{(s)} + \mathcal{N}(f^{(s)})$ where $\mathcal{L} f^{(s)} = -v\partial_x f^{(s)} - \kappa^{(s)} E\, \partial_v F_0^{(s)}$ and $\mathcal{N}(f^{(s)}) = -\kappa^{(s)} E\, \partial_v f^{(s)}$.

An unstable mode exists if the dielectric function

$$\epsilon_k(z) \equiv 1 - \frac{1}{k^2} \int_{-\infty}^{\infty} dv\, \frac{\sum_s \kappa^{(s)} \partial_v F_0^{(s)}(v)}{v - z}, \quad (\mathrm{Im}\, z > 0) \tag{5}$$

has a root $z_0 = v_p + i\gamma/k$ in the upper half plane ($\gamma > 0$). The root determines an eigenvalue $\lambda = -ikz_0$ for $\mathcal{L}$ with a two-component eigenvector

$$\Psi = e^{ikx} \begin{pmatrix} \psi^{(e)}(v) \\ \psi^{(i)}(v) \end{pmatrix}. \tag{6}$$

We assume that there is a single such mode and that it corresponds to a simple root of $\epsilon_k(z)$, i.e., $\epsilon_k'(z_0) \neq 0$. The eigenvalue $\lambda$ can be real or complex depending on the equilibrium; the ion-acoustic instability corresponds to a complex eigenvalue.

The amplitude of the unstable mode is the coefficient of $\Psi$ in the expansion of $f$

$$f(x,v,t) = [A(t)\Psi(x,v) + \text{c.c.}] + S(x,v,t); \tag{7}$$

here $f$ denotes the two-component field $f \equiv (f^{(e)}, f^{(i)})$ and represents the full nonlinear solution. This decomposition allows the dynamics of the mode amplitude $A(t)$ and the remaining modes $S(x,v,t)$ to be separated

$$\dot{A} = \lambda A + (\tilde{\Psi}, \mathcal{N}(f)), \tag{8}$$

$$\partial_t S = \mathcal{L} S + \mathcal{N}(f) - [(\tilde{\Psi}, \mathcal{N}(f))\,\Psi + \text{c.c.}], \tag{9}$$

using an inner product defined for two-component fields $B = (B^{(e)}, B^{(i)})$ and $D = (D^{(e)}, D^{(i)})$ by $(B,D) \equiv \int dx \int dv\, [B^{(e)}(x,v)^* D^{(e)}(x,v) + B^{(i)}(x,v)^* D^{(i)}(x,v)]$, and the adjoint eigenfunction $\tilde{\Psi}$ for $\lambda^*$. In (8) and (9) the linear terms are decoupled, but nonlinear couplings between $\dot{A}$ and $\partial_t S$ remain.

The amplitude equation for $A$ follows when we express the time dependence of $S$ in terms of $A$: $S(x,v,t) = H[x,v,A(t),A^*(t)]$. As we have discussed elsewhere, this step can be visualized as a restriction of the initial condition to the two-dimensional unstable manifold of the equilibrium [16]. Consistency between the time dependence of $S = H$ and the evolution of $S$ described by (8) and (9) requires that $H[x,v,A(t),A^*(t)]$ satisfy

$$[\dot{A}\, \partial_A H + \dot{A}^*\, \partial_{A^*} H]|_{f=f^u} = \mathcal{L} H + \mathcal{N}(f^u) - [(\tilde{\Psi}, \mathcal{N}(f^u))\,\Psi + (\tilde{\Psi}, \mathcal{N}(f^u))^*\,\Psi^*], \tag{10}$$

where $f^u(x,v) = [A\Psi(x,v) + \text{c.c.}] + H(x,v,A,A^*)$. For solutions of this form, the dynamics of $A(t)$ (8) yields an autonomous equation for $A$,

$$\dot{A} = \lambda A + (\tilde{\Psi}, \mathcal{N}(f^u)), \tag{11}$$

provided $H$ can be determined from (10). The homogeneity of the equilibrium $F_0$ forces this amplitude equation to have a simple form: $\dot{A} = A\, p(|A|^2)$ where $p(|A|^2)$ must still be determined. In polar variables, $A = \rho e^{-i\theta}$, the system (11) separates

$$\dot{\rho} = \rho\, \mathrm{Re}\,[p(\rho^2)], \qquad \dot{\theta} = -\mathrm{Im}\,[p(\rho^2)], \tag{12}$$

yielding a one-dimensional flow for $\rho(t)$; the essential problem is to study $p(\rho^2)$.

Our conclusions regarding the evolution of the wave are based on an analysis of the amplitude expansion for $p$,

$$p(\rho^2) = \sum_{j=0}^{\infty} p_j\, \rho^{2j}, \tag{13}$$

and similar expansions for $H(x,v,A,A^*)$. By substituting $\dot{A} = A\, p(\rho^2)$ into (11) we obtain one set of relations between the coefficients of $p$ and $H$: $A \sum_j p_j\, \rho^{2j} = \lambda A + (\tilde{\Psi}, \mathcal{N}(f^u))$; the defining equation (10) for $H$ provides a second set of relations. The expansion coefficients are calculated by solving these coupled relations recursively; full details of this calculation and our analysis of the resulting recursion relations will be published elsewhere [17].

The first coefficient $p_0$ is simply the linear eigenvalue $\lambda$. Information about the nonlinear evolution is provided by the higher order coefficients; in particular, the first nonlinear term is given by

$$p_1 = \frac{1}{\gamma^4}\left[\frac{\kappa^{(i)}(1-\kappa^{(i)2})\ \mathrm{Im}(\alpha(z_0))}{4k\epsilon_k'(z_0)} - \gamma\left(\frac{1}{4} + \frac{\kappa^{(i)}(1-\kappa^{(i)2})\alpha'(z_0)}{4k^2\epsilon_k'(z_0)}\right) + \mathcal{O}(\gamma^2)\right], \quad (14)$$

where $\alpha'(z)$ is the derivative of $\alpha(z) \equiv \int dv\, \partial_v F_0^{(i)}/(v - z)$. If

$$\kappa^{(i)}(1 - \kappa^{(i)2})\ \mathrm{Im}(\alpha(z_0)) \neq 0\,, \quad (15)$$

then the first term in (14) is nonzero and, in the limit of weak instability $\gamma \to 0^+$, there is a $\gamma^{-4}$ singularity in $p_1$. When (15) holds, the trapping scaling introduced previously for $\rho$ no longer removes the singularity in $p_1$, and a new scaling $\rho(t) = \gamma^{5/2}\, r(\gamma t)$ is required to obtain a nonsingular leading order term.

Beyond this leading order calculation, we have proved that the coefficients in (13) have the asymptotic behavior $p_j \sim \gamma^{-(5j-1)}$ to all orders $j \geq 1$. This general result implies that when Eqs. (12) are rewritten in the rescaled amplitude $r$ the singularities will be canceled to *all* orders in $r$ [17]. Thus given the generic condition (15) the theory predicts the asymptotic scaling for the mode electric field is $|E_k| \sim \gamma^{5/2}$ as $\gamma \to 0^+$.

Our prediction must be reexamined if the genericity condition (15) fails and the leading term in $p_1$ vanishes. The three factors in (15) indicate three exceptional circumstances when this can happen: $\kappa^{(i)} = 0$, $\mathrm{Im}(\alpha(z_0)) = 0$, or $\kappa^{(i)2} = 1$. The first circumstance corresponds to the limit of fixed ions, $m_e/m_i \to 0$, and the cubic term (14) reduces to our previous result (3). With fixed ions, the terms of the amplitude expansion are known to be nonsingular to all orders once the amplitude has been scaled with $\gamma$ to balance the divergence in $p_1$, i.e., $\rho(t) = \gamma^2\, r(\gamma t)$ [16].

The second circumstance arises naturally when the equilibria under consideration are reflection symmetric $F_0^{(s)}(-v) = F_0^{(s)}(v)$. With such a reflection symmetry, one can find pure imaginary roots $z_0^* = -z_0$ (hence real eigenvalues) and then $\alpha(z_0)$ is forced to be real; an example is provided by a reflection-symmetric two-stream instability. In this case, the trapping scaling for $\rho$ cancels the singularity in $p_1$ and moreover can be proved to yield an amplitude expansion which is nonsingular to all orders [17]. As in the fixed ion case, this result predicts the familiar trapping scaling for the mode electric field $|E_k(t)| \sim \gamma^2$.

The third exceptional circumstance $\kappa^{(i)2} = 1$ requires $(q_i m_e/m_i)^2 = 1$ which corresponds to an electron-positron plasma ($q_i = 1$, $m_i = m_e$). In this case the singularity structure of the expansion is more complicated. The cubic coefficient has a $\gamma^{-3}$ singularity which suggests trapping scaling for $\rho$, but the divergence of fifth order coefficient turns out to be $p_2 \sim \gamma^{-8}$ which is *not* removed by trapping scaling [17]. This fifth order singularity is canceled if we set $\rho(t) = \gamma^{9/4}\, r(\gamma t)$. However, inspection of the higher order terms in the expansion shows the divergence structure of $p_j \sim \gamma^{-(5j-3)}$, and canceling these singularities to all orders again requires the generic scaling $\rho(t) = \gamma^{5/2}\, r(\gamma t)$ even though to any finite order $p_j$ a smaller exponent would suffice [17].

With the notable exception of this third example, our conclusions are easily summarized: the scaling required to obtain a nonsingular expansion is correctly predicted by the divergence found in the cubic coefficient $p_1$. Generically, this singularity dictates a scaling by $\gamma^{5/2}$, but this is replaced by $\gamma^2$ in the limit of infinite ion inertia or for instabilities in reflection-symmetric systems due to real eigenvalues. In the generic case we can estimate the range of growth rates where the new $\gamma^{5/2}$ scaling is visible by determining the range in $\gamma$ where the $\gamma^{-4}$ divergence dominates $p_1$. In general, this will depend on the specific parameters of the equilibrium, and we give two illustrative examples: an unstable plasma wave driven by an electron beam and an ion-acoustic instability. For simplicity, we have evaluated the dispersion relation and the cubic coefficient using Lorentzian distributions for the two species,

$$F_0^{(e)}(v) = \frac{-\alpha_e\, n_p/\pi}{(v - u_e)^2 + \alpha_e^2} - \frac{\alpha_b\, n_b/\pi}{(v - u_b)^2 + \alpha_b^2}\,, \qquad F_0^{(i)}(v) = \frac{\alpha_i/\pi}{v^2 + \alpha_i^2}\,, \quad (16)$$

with $n_p + n_b = n_i = 1$.

A plasma wave instability is calculated for an equal density cold beam with four mass ratios; the parameters of this example are $2n_p = 2n_b = n_i = 1$, $\alpha_e = \alpha_i = 10\alpha_b = 1.0$, and $u_e = 0$. The variation of $\gamma^3\ \mathrm{Re}(p_1)$ as $\gamma \to 0^+$ is shown in Fig. 1. For each mass ratio in Fig. 1, we fix $k$ and vary $u_b$; the chosen values are $k = 0.75$ for $m_e/m_i = 0.5$ and $k = 0.5$ for $m_e/m_i = 0.1, 0.01$, and $0.001$. As $m_e/m_i$ decreases, the fixed ion result $\gamma^3\ \mathrm{Re}(p_1) \to -1/4$ holds down to smaller and smaller growth rates. Interestingly, the effect of the mobile ions is to shift the asymptotic sign of $\mathrm{Re}(p_1)$ from negative to positive. Since the higher order terms in the amplitude expansion cannot be neglected, this does not automatically imply the onset of subcritical bifurcation but may nevertheless be significant.

An ion-acoustic instability is shown in Fig. 2 also for four mass ratios. In this case $\alpha_e = 100\alpha_i = 1.0$, $n_b = 0$, and there is only a drifting electron population with $n_p = n_i = 1$. For each mass ratio, we fix $k$ and vary $u_e$; the chosen values are $k = 1.5, 1.2, 0.75$, and $0.4$ corresponding to $m_e/m_i = 0.5, 0.1, 0.01$, and $0.001$, respectively. Here the effect of decreasing $m_e/m_i$ is to

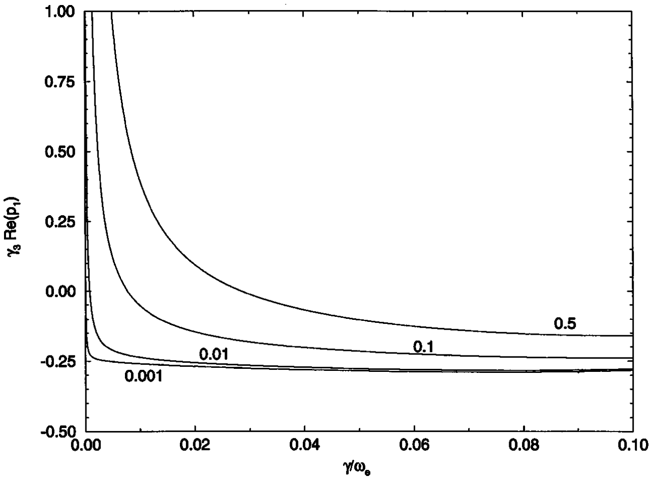


FIG. 1. Asymptotic behavior of $\gamma^3\,\mathrm{Re}(p_1)$ for the beam-plasma instability; each curve is labeled by mass ratio. The growth rate $\gamma$ is measured in units of the electron plasma frequency. The divergence as $\gamma \rightarrow 0^+$ indicates the regime predicted to show the generic scaling $|E_k| \sim \gamma^{5/2}$.

increase the range of the generic scaling as measured in units of the ion plasma frequency $\omega_i^2 = 4\pi e^2 n_i/m_i$.

The qualitative result of the scaling $|E_k| \sim \gamma^{5/2}$ is to reduce the electric field of the nonlinearly saturated wave compared to a wave characterized by trapping scaling. The fact that the new singularities disappear if the ion inertia becomes infinite strongly indicates that the ion response is key. For small $\gamma$, near the resonant velocity, the finite mass ions see a nearly steady, very slowly growing electric field, and they can absorb energy from the wave (not possible for fixed ions). The available kinetic energy of the drifting electrons is now presumably *shared* by the wave and the mobile ions with the result that the saturated wave has a smaller amplitude. We do not understand this sharing well enough to provide an intuitive derivation of the $\gamma^{5/2}$ scaling. In addition, the fact that the effect disappears when the equilibria are reflection symmetric suggests there are subtleties not accounted for by the simple observation that the ions can absorb energy. More work is needed to understand how the mobile ions manage to cut off the growth of the unstable mode at such small amplitudes.

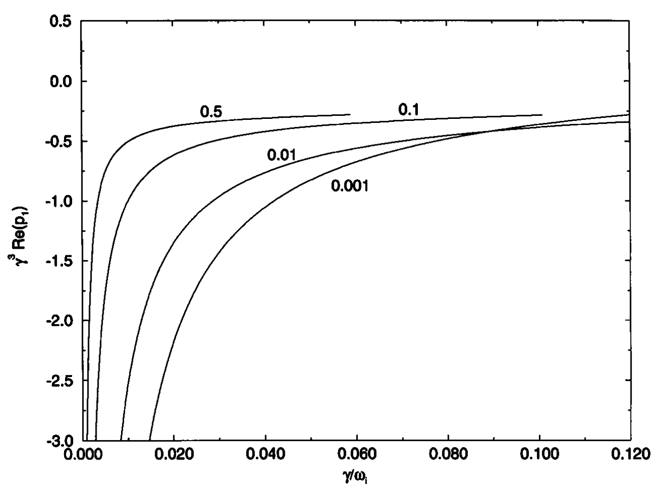


FIG. 2. Asymptotic behavior of $\gamma^3\ \mathrm{Re}(p_1)$ for an ion-acoustic instability. The growth rate $\gamma$ is measured in units of the ion plasma frequency. The divergence as $\gamma \rightarrow 0^+$, indicates the regime predicted to show the generic scaling $|E_k| \sim \gamma^{5/2}$.

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