New Amplitude Equations for Thin Elastic Rods

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The stability of twisted straight rods is described within the framework of the time dependent Kirchhoff equations for thin elastic filaments. A perturbation method is developed to study the linear stability of this problem and find the dispersion relations. A nonlinear analysis results in a new amplitude equation, describing the deformation of the rod beyond the instability, which takes the form of a pair of nonlinear, second-order evolution equations coupling the local deformation amplitude to the twist density. Various solutions, such as solitary waves, are presented. [S0031-9007(96)01453-6]

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Filaments are fundamental physical structures that can be found in many guises on many different scales. They appear in various problems in biology, chemistry, physics, and engineering [1]. One way to model these structures is to assume that they are made of an elastic material obeying the appropriate laws of elasticity. The Kirchhoff model for rods describes the dynamics of (thin) elastic filaments within the approximation of linear elasticity theory. The stability of stationary filaments under external (or internal) constraints is one of the oldest fundamental problems in the theory of elasticity, dating back to Euler. The general problem is to understand and describe the postbifurcation behavior of stationary structures. To answer this question, many authors have considered the linear (and nonlinear) analysis of phenomenological equations such as the (one-dimensional) "beam equation" [2]. However, these simplified equations lack crucial information on the three-dimensional structure of the solution after bifurcation. In addition, these analyses have mainly been confined to stationary perturbations.

Here, we study the dynamical (i.e., time-dependent) stability of the full three-dimensional Kirchhoff equations for the twisted straight rod. We first develop a new perturbation scheme to study the stability of stationary solutions. These perturbation expansions are performed at the level of a local basis (the director basis) attached to the central axis of the curve. We use these expansions to perform both linear and nonlinear analyses with the latter leading to a new amplitude equation describing how the rod deformation amplitude is coupled to the twist density for the solutions after, but close to, bifurcation.

We first consider a simple space curve, x, parametrized by arc length, s, whose position may vary in time, i.e., $x = x(s,t)$. We assume that x is at least twice differentiable. In what follows, $()'$ denotes differentiation with respect to s and $\dot{()}$ differentiation with respect to time. At each point of the curve, one can define a *local orthonormal basis* $d_i = d_i(s,t)$, $i = 1,2,3$, by introducing the tangent vector $d_3 = x'(s,t)$ and choosing two unit vectors d_1, d_2 in the plane normal to d_3 such that (d_1, d_2, d_3) forms a right-handed orthonormal basis for each value of (s,t). By construction there exist a *twist vector* $\kappa = \kappa_1 d_1 + \kappa_2 d_2 + \kappa_3 d_3$ and a *spin vector* $\omega = \omega_1 d_1 + \omega_2 d_2 + \omega_3 d_3$ which control the space and time evolution of the basis along the curve via the *spin and twist equations*,

$$d_i' = \kappa \times d_i, \qquad \dot{d}_i = \omega \times d_i, \qquad i = 1,2,3. \quad (1)$$

Knowledge of $\kappa = \kappa(s,t)$ and $\omega = \omega(s,t)$ is enough to construct the position and motion of the curve in space (up to a rigid translation), since the solution of the spin and twist equations determines $d_3 = d_3(s,t)$ which can be integrated once to give x. If one chooses d_1 to be the normal vector to the curve, then d_2 is the binormal and the local basis reduces to the well-known Frenet frame [3].

The Kirchhoff model of rod dynamics considers rods whose length is much greater than the cross sectional radius. Moreover, here it is assumed that the rod is inextensible and of circular cross section. These additional assumptions are not essential to the procedure described below and could be relaxed if required. A one-dimensional theory can be derived in which all the relevant physical quantities are averaged over the cross sections and attached to the central axis. It follows that the total force $F = F(s,t)$ and the total moment $M = M(s,t)$ can be expressed locally in terms of the director basis, i.e., $F = \sum_{i=1}^{3} f_i d_i$, $M = \sum_{i=1}^{3} M_i d_i$. The conservation of linear and angular momentum then leads to equations for the force and the moment which, in appropriately scaled variables, take the form [4,5],

$$F'' = \ddot{d}_3, \quad (2)$$

$$M' + d_3 \times F = d_1 \times \ddot{d}_1 + d_2 \times \ddot{d}_2, \quad (3)$$

$$M = \kappa_1 d_1 + \kappa_2 d_2 + \Gamma \kappa_3 d_3. \quad (4)$$

The last equation is the constitutive relationship of linear elasticity theory. It introduces the elastic parameter $\Gamma = 1/(1 + \sigma)$ (where σ is the Poisson ratio) which measures the ratio between bending and twisting coefficients of the rod. Typically, Γ varies between $2/3$ and 1.

Assuming that a stationary solution of these equations is known, we can study its stability (with respect to different parameters) by looking for the parameter values at which new, time-dependent, solutions exist. In order to do so, we must consider a perturbation expansion of the variables. We demand that to each order the approximated local basis ($d_i = d_i^{(0)} + \epsilon d_i^{(1)} + \epsilon^2 d_i^{(2)} + \cdots$) remains orthonormal. That is, to order $O(\epsilon^m)$, we must have $d_i \cdot d_j = \delta_{ij} + O(\epsilon^{m+1})$. This constraint introduces at each order three arbitrary parameters ($\alpha_1^{(m)}, \alpha_2^{(m)}, \alpha_2^{(m)}$) that allow us to express the perturbed basis in terms of the unperturbed one in the following way [6]:

$$d_i^{(m)} = \alpha^{(m)} \times d_i^{(0)} + \sum_j \beta_{ij}^{(m)} d_j^{(0)}, \qquad i = 1, 2, 3, \quad (5)$$

where $\beta^{(m)}$ is a symmetric tensor whose entries depend only on $\alpha^{(k)}$ with $k < m$.

The perturbation expansion of the twist and spin vector can be expressed in terms of α and the unperturbed twist and spin vectors,

$$\kappa = \kappa^{(0)} + (\alpha^{(1)})' + \kappa^{(0)} \times \alpha^{(1)} + O(\epsilon^2), \quad (6)$$

$$\omega = \dot{\alpha}^{(1)} + O(\epsilon^2). \quad (7)$$

The force F can also be expanded in ϵ,

$$F = \sum_i \{f_i^{(0)} + \epsilon[f_i^{(1)} + (\alpha \times f^{(0)})_i]\} d_i^{(0)} + O(\epsilon^2). \quad (8)$$

Higher order terms in all these expansions can easily be generated.

The unperturbed stationary configuration is characterized by $(f^{(0)}, \kappa^{(0)})$. Using the perturbation scheme described here, we can obtain the appropriate *variational equations*, i.e., the linearization of Eqs. (2) and (3) about the exact (stationary) solution. These are written in the form

$$L_E(\kappa^{(0)}, f^{(0)}) \cdot \mu^{(1)} = 0, \quad (9)$$

where L_E is a linear, second-order differential operator in s and t whose coefficients depend on s through the unperturbed solution $\kappa^{(0)}, f^{(0)}$ and $\mu^{(1)}$ is the six-dimensional vector $\mu^{(1)} = \{\alpha^{(1)}, f^{(1)}\}$. The explicit form of this linear system of equations is given in [6].

In the case of the twisted straight rod we choose the local basis in such a way that the vectors (d_1, d_2) follow the twist γ. In these variables, the stationary solution can be written

$$\kappa^{(0)} = (0, 0, \gamma), \qquad f^{(0)} = (0, 0, P^2). \quad (10)$$

We consider here a rod under tension $[f_3^{(0)} > 0]$ rather than under compression $[f_3^{(0)} < 0]$.

The linear solutions [to (9)] can be expressed as

$$\mu_j^{(1)} = e^{\sigma t}(A x_j e^{ins} + A^* x_j^* e^{-ins}), \qquad j = 1, \ldots, 6, \quad (11)$$

where the growth rate σ is determined from the dispersion relations, $\Delta(\sigma, n) = 0$, obtained by substituting (11) into (9). The neutral curves correspond to the values of the parameters for which the stationary solutions bifurcate to give new solutions. They are obtained by considering the solution of $\Delta(0, n) = 0$, namely,

$$(\gamma^2 - n^2)\{[\gamma^2(\Gamma - 1) - P^2 - n^2]^2 - \gamma^2(\Gamma - 2)^2 n^2\} = 0. \quad (12)$$

A typical plot of these relations is shown in Fig. 1. The straight line portion of this figure does not correspond to an actual solution. Indeed, after reconstruction of the rod, the mode $n = \gamma$ gives the trivial (null) solution (i.e., $\kappa^{(1)} = \omega^{(1)} = 0$). The parabolic neutral curve corresponds to an *unstable* helix with critical parameters $n_c = P(2 - \Gamma)/\Gamma$ and $\gamma_c = \pm 2P/\Gamma$. For fixed P, the straight rod becomes unstable at the critical twist γ_c and the new shape is helicoidal, taking the form

$$x = \left(s, -\frac{2A}{P}\sin sP, \frac{2A}{P}\cos sP\right). \quad (13)$$

In addition, we note the presence of the neutral mode $n = 0$ which corresponds to an arbitrary rotation about the central axis and an arbitrary increase of the tension.

The linear analysis can only describe the situation at threshold. In order to describe the evolution of the unstable modes beyond this point the effect of nonlinearities must be included through an appropriate nonlinear analysis [7]. This is achieved by expanding about the critical twist γ_c and involves introducing the perturbation parameter

$$\epsilon^2 = \gamma - \gamma_c, \quad (14)$$

and the stretched time and space scales $t_1 = \epsilon t$ and $s_1 = \epsilon s$. To order $0(\epsilon)$, the (linear) solution is given by a superposition of the neutral modes, namely,

$$\mu^{(1)} = Y\xi_0 + X\xi_n e^{ins} + X^* \xi_n^* e^{-ins},$$

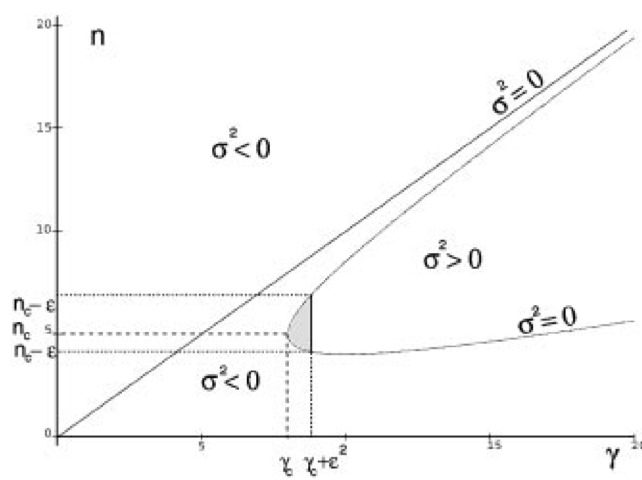

FIG. 1. Dispersion relation for $P = 3$, the straight line does not correspond to new solutions, but the parabola is the neutral curve defining the unstable helix. The line $n = 0$ is a possible neutral solutions corresponding to an arbitrary twist.

where $Y = Y(s_1, t_1)$ and $X = X(s_1, t_1)$ represent, respectively, the slowly varying amplitudes of the axial twist and the unstable helical mode; $n = n_c$; and

$$\xi_0 = (0, 0, 1, 0, 0, 1), \qquad \xi_n = (1, i, 0, -iP^2, P^2, 0). \tag{15}$$

At this order of ϵ the functions Y and X are arbitrary and constant but may vary on the longer scales (s_1, t_1).

The amplitude equations describing the slow evolution of the rod on the stretched scales s_1, t_1 are derived by the method of multiple scales analysis. To third order in ϵ, an equation for the amplitudes (Y, X) can be derived by requiring that the solutions remain bounded in space which leads to a Fredholm alternative condition. The final result is two equations for the amplitudes Y, X [7],

$$\left(\frac{P^2 + 1}{P^2}\right)\frac{\partial^2 X}{\partial t_1^2} - \frac{\partial^2 X}{\partial s_1^2} = P\Gamma X\left(1 - 2P|X|^2 + \frac{\partial Y}{\partial s_1}\right),$$
$$\frac{2}{\Gamma}\frac{\partial^2 Y}{\partial t_1^2} - \frac{\partial^2 Y}{\partial s_1^2} = -2P\frac{\partial |X|^2}{\partial s_1}. \tag{16}$$

In this coupled system of equations the twist density Y plays a central role. If we set $Y = 0$ it is easy to see that the stationary solutions may blow up in finite space. The amplitude equations can also be understood in terms of symmetry breaking [8] in that the first-order derivatives with respect to s_1 break the symmetry associated with the rotation of the rod about the central axis and as a result introduce a twist-imposed handedness in the postbifurcation solution.

Although these equations (16) are probably nonintegrable (they fail the Painlevé test [9] for partial differential equations) the following interesting special solutions can be obtained:

(a) Homogeneous solutions. The spatially independent form of (16) is simply

$$\frac{\partial^2 X}{\partial t_1^2} = \frac{P^3\Gamma}{P^2 + 1}X(1 - 2P|X|^2), \tag{17}$$

where the twist density decouples from the deformation and is set equal to a constant. After integration, the filament solution is found to correspond to a helix,

$$x = \left(s, -\frac{2\epsilon X(\epsilon t)}{P}\sin sP, \frac{2\epsilon X(\epsilon t)}{P}\cos sP\right). \tag{18}$$

(b) Traveling wave solutions. Setting $z = s_1 - ct_1$ one obtains the traveling wave reduction of the amplitude equations which, after simplification, take the form

$$\frac{\partial^2 X}{\partial z^2} = \frac{P^3\Gamma}{P^2(c^2 - 1) + c^2}X\left(K + 1 + \frac{4Pc^2}{\Gamma - 2c^2}|X|^2\right),$$

where K is an arbitrary constant chosen in such a way that the derivative of the twist goes to zero at infinity.

For $c^2 > \Gamma/2$, two interesting situations arise. If $c^2 > \max\{\Gamma/2, P^2/(P^2 + 1)\}$ a homoclinic orbit can be found,

$$X(z) = \rho_1 \mathrm{sech}(\rho_2 z). \tag{19}$$

with $K = 0$, $\rho_1^2 = (2c^2 - \Gamma)/(2Pc^2)$, $\rho_2^2 = (P^3\Gamma)/(P^2c^2 - P^2 + c^2)$, which corresponds to a *pulselike* solitary wave solution traveling along the rod with constant speed c. For $P^2/(P^2 + 1) > c^2 > \Gamma/2$, one finds a heteroclinic connection of the form

$$X(z) = \rho_1 \tanh(\rho_2 z), \tag{20}$$

with $K = \Gamma/(\Gamma - 2c^2)$, $\rho_1^2 = 1/2P$, and $\rho_2^2 = c^2P^3\Gamma/[(P^2c^2 - P^2 + c^2)(\Gamma - 2c^2)]$, which describes a *frontlike* solitary wave connecting two different asymptotic states. The two different solutions are shown in Fig. 2.

We remark that the minimum speed of these traveling waves is $c^2 = \Gamma/2$. This is the speed of the torsional waves obtained from elementary linear elastic theory [10]. We believe that the solutions obtained here are the nonlinear version of torsional waves that takes into account the three-dimensional structure of the system and allows propagation of waves between regions with different twist densities.

The propagation of solitary waves along an elastic rod has been of interest in recent years [5,11] and some particular exact traveling wave solutions have been obtained for systems with constant twist. The solutions presented here are obtained as possible postbifurcation behaviors of the twisted straight rod and have nonconstant twist density Y. Preliminary numerical results show that some of these solutions are remarkably stable and that there exists a mechanism for selection between different solutions with different speeds. Also, the collisions between pairs of stable pulses with opposite, but equal, speeds show near perfect (i.e., form preserving) collisions. However, closer inspection of the solutions show that there is a small amount of radiation loss clearly indicating that the system is not completely integrable [12].

(c) Stationary solutions. The stationary limit of (16) reduces, somewhat remarkably, to a simple system whose

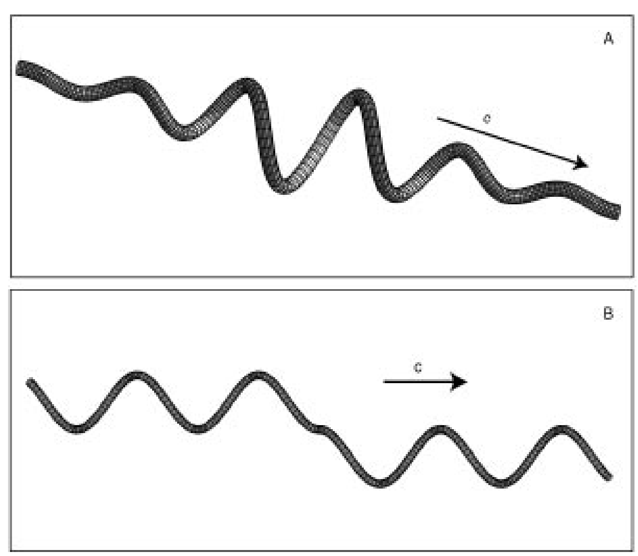

FIG. 2. Traveling waves solutions: The pulse (a) $c = 4$, $P = 7$ and front solutions (b} $c = 0.7$, $P = 1$. $\Gamma = 3/4$ in both cases.

equation for X is linear. The solutions are easily obtained,

$$X(s_1) = K_2 e^{i\sqrt{P\Gamma K_1}\,s_1} + K_3 e^{-i\sqrt{P\Gamma K_1}\,s_1},$$
$$Y(s_1) = K_2 K_3^* e^{2i\sqrt{P\Gamma K_1}\,s_1} + K_2^* K_3 e^{-2i\sqrt{P\Gamma K_1}\,s_1} + s_1[1 - K_1 - 2P(|K_2|^2 + |K_3|^2)] + K_4,$$

where the K_i are constants determined by the boundary conditions.

If we hold the extremities of a finite rod of length L fixed, the constants K_i are determined and we find an envelope solution for the rod,

$$x(s) = \left(s, \frac{\sqrt{4L^2\epsilon^2 P\Gamma - 1}}{2PL\sqrt{\Gamma}a_+a_-} \times (a_- \cos a_+ s - a_+ \cos a_- s), \frac{\sqrt{4L^2\epsilon^2 P\Gamma - 1}}{2PL\sqrt{\Gamma}\,a_+a_-} \times (a_- \sin a_+ s - a_+ \sin a_- s)\right),$$

where $a_\pm = P \pm 1/2L$. This solution is shown in Fig. 3 for a particular set of parameters.

As described in [7] this solution reveals a delay in the bifurcation as a function of the rod length. Indeed, the bifurcation now occurs at $\gamma = \gamma_c + 1/4P\Gamma L^2$. A similar influence of the boundary conditions on bifurcation has been described in fluid dynamics [13].

The amplitude equations (16) are of interest in their own right and deserve more theoretical and numerical study. Preliminary work reveals that they have a Hamiltonian structure. The stability and selection mechanisms of different solutions will be described elsewhere [12]. The Kirchhoff equations are a central model to study the different conformations of filamentary structures. Most of the theoretical work has focused on obtaining stationary solutions. There is now a vast body of literature concerning this problem. However, the dynamics of solutions after bifurcation has, to the best of our knowledge, hardly been investigated. The methods we have developed to study the dynamical stability of stationary solutions are quite general and have been applied to different structure such as the twisted ring and the helix. These studies reveal an extraordinarily rich dynamical behavior of filaments after bifurcations, and we hope they will be further developed to enrich our understanding of these universal structures.

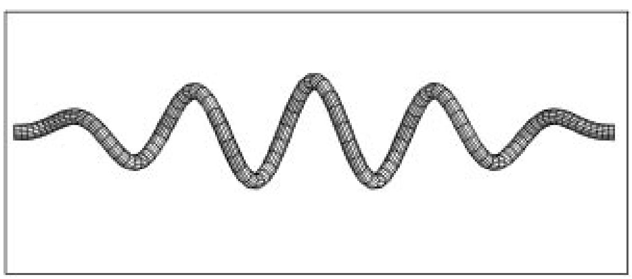

FIG. 3. The stationary solution for ($P = 5$, $\Gamma = 3/4$).

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