Analytic Three-Dimensional Solutions of the Magnetohydrostatic Equations with Twisted Field Lines

R. Kaiser
Universität Bayreuth, Lehrstuhl für Angewandte Mathematik, 95440 Bayreuth, Germany

A. Salat
Max-Planck-Institut für Plasmaphysik, EURATOM Association, Boltzmannstrasse 2, 85748 Garching bei München, Germany
(Received 12 April 1996)

Two types of exact solutions of the MHD equilibrium equations are presented. The equilibria exhibit neither continuous nor mirror symmetry. The configurations are infinitely extended along a straight axis with surfaces of constant pressure closed around the axis. The cross sections are elliptic for equilibria of the first type. For the second type they are elliptic only close to the axis. Field and current lines turn around the axis and extend from minus to plus infinity in the axial direction. In these limits the configurations become singular. The rotational transform and the local shear are discussed. [S0031-9007(96)01351-8]

PACS numbers: 52.30.Bt, 52.55.Dy, 52.55.Hc

The most promising way to stably confine a plasma sufficiently long in order to achieve thermonuclear fusion is magnetic confinement in a toroidal configuration. Two such configurations have been investigated especially carefully. In the tokamak a toroidal magnetic field generated by external coils together with an externally driven toroidal current sustain an, in principle, axially symmetric equilibrium. The discrete set of coils, however, leads to an unavoidable "rippling" of the magnetic field and thus to a certain amount of nonsymmetry. A stellarator, in contrast, is an inherently three-dimensional (3D) configuration far from axisymmetry, whose magnetic field is generated completely by external coils. In both configurations the magnetic field lines twist around the axis with finite pitch. The equilibria are supposed to be well described by the equations of ideal MHD,

$$\begin{aligned} \mathbf{j}\times\mathbf{B} &= \nabla P\,, \\ \mathbf{j} &= \nabla\times\mathbf{B}\,, \\ \nabla\cdot\mathbf{B} &= 0\,, \end{aligned} \tag{1}$$

where $\mathbf{B}$, $\mathbf{j}$, and P denote the magnetic field, current density, and pressure, respectively.

Knowledge of analytic nonsymmetric toroidal equilibria, if they exist, would bring many benefits; the existence problem would be settled; analytic equilibria would be at hand to do quick parameter studies; wave propagation, for diagnostic and heating purposes, could be studied on a safe basis; the effect of small chaotic regions on transport would lose its urgency; and numerical equilibrium codes could be tested.

Note that Eqs. (1) describe yet another situation of physical interest, viz. steady flow of an inviscid, incompressible fluid of unit density. In that case $\mathbf{B}$, $\mathbf{j}$, and P denote the velocity field $\mathbf{v}$, its vorticity, and (up to a constant) the Bernoulli function $-(p + v^2/2)$, respectively (see, e.g., [1]).

To find 3D solutions of Eqs. (1) is, however, notoriously difficult. This is because Eqs. (1) represent a system of partial differential equations of mixed elliptic-hyperbolic type, for which there is no general theory on existence and uniqueness of solutions. Only with additional assumptions can the problem be rendered tractable. For plane, axial, or helical symmetry Eqs. (1) can be reduced to a single quasilinear elliptic equation for which an elaborate existence theory as well as many explicit solutions are known [2–4]. The axisymmetric case, for example, is governed by the well-known Grad-Shafranov equation [5].

Discrete symmetries seem to facilitate the problem too. In [6] equilibria with a purely toroidal magnetic field, which is mirror symmetric with respect to a poloidal plane have been considered. In that particular case the so-called β iteration [7] was proved to converge for low plasma pressure. The pertinent pressure surfaces, however, are not poloidally closed in general.

If symmetries are no longer present in the system, numerical work shows that magnetic surfaces tend to disrupt and magnetic islands and ergodic regions appear in Poincaré plots of the field lines (see, e.g., [8]). A well-defined smooth pressure function may not be expected in these cases. This view is corroborated by theoretical analysis of Eqs. (1) for nonaxisymmetric equilibria with field line shear [3,9]. It indicates that finite jumps in the pressure distribution and the formation of current sheets at each rational surface are to be expected generically. Grad conjectured for that reason that nonsymmetric equilibria in the sense of classical solutions do not exist at all [9].

It is due to these intricacies that the present-day available exact results on existence and nonexistence of equilibria refer to quite special configurations. *Nonexistence* of configurations with purely poloidal magnetic fields, for example, has been investigated in [10,11]. Other nonexistence results refer to the isodynamic [12] or the "quasi-helical" stellarator [13].

As to the *existence* of 3D equilibria all results [14–17] refer to straight-axis configurations (with the exception of the mirror-symmetric configuration mentioned above). If such configurations are poloidally closed and periodic in the axial direction, they are still topologically equivalent to a torus. The solutions obtained in [14–16] are of this type, in the sense that free functions along the axis can be chosen to be periodic. The field lines, however, are confined there to planes orthogonal to the axis. The solutions in [17], on the other hand, have field lines purely along the axis. Thus, in neither of these cases do the field lines twist around the axis with a finite slope.

In contrast to these previous results, the equilibria presented here are—to our knowledge—the first examples of analytic 3D equilibria with twisted field lines. They demonstrate that field line twist need not be detrimental to 3D configurations as was suspected earlier [9].

Two types of solutions of Eqs. (1) were obtained. In both cases the magnetic field has the representation

$$\mathbf{B}(x,y,z) = \nabla\times[H(x,y,z)\nabla z] + \nabla\times\nabla\times[K(x,y,z)\nabla z]. \tag{2}$$

For the first type the functions H and K are given by

$$\begin{aligned} H(x,y,z) &= \frac{-h}{2}[a(z)x^2 + 2b(z)xy + c(z)y^2], \\ K(x,y,z) &= \frac{-k}{2\kappa}[a(z)x^2 + 2b(z)xy + c(z)y^2], \end{aligned} \tag{3}$$

while the pressure P is determined by

$$\begin{aligned} P^*(x,y,z) \equiv & -2\frac{P - P_0}{h^2 + k^2} \\ &= [a(z) + c(z)][a(z)x^2 \\ &\quad + 2b(z)xy + c(z)y^2]. \end{aligned} \tag{4}$$

Here, h and k are two arbitrary constants which determine the weight of the contributions from H and K, respectively, to the magnetic field. P_0 is the pressure on the z axis. The functions $a(z), b(z), c(z)$ are as follows:

$$\begin{aligned} a(z) &= a_1 e^{\kappa z} + a_2 e^{-\kappa z}, \\ b(z) &= b_1 e^{\kappa z} + b_2 e^{-\kappa z}, \\ c(z) &= c_1 e^{\kappa z} + c_2 e^{-\kappa z}, \end{aligned} \tag{5}$$

with arbitrary constants $\kappa, a_1, a_2, c_1, c_2$. The coefficients b_1, b_2 are related to the other ones by

$$b_1^2 = a_1 c_1, \qquad b_2^2 = a_2 c_2. \tag{6}$$

A consequence thereof is the relation

$$a(z)c(z) - b^2(z) = (\sqrt{a_1 c_2} - \sqrt{a_2 c_1})^2 = \text{const} \geq 0. \tag{7}$$

The functions a, b, c can be normalized by setting the constant in the equation above to 1.

The surfaces of constant pressure are closed surfaces, composed of ellipses nested around the z axis. At fixed P their width, axis ratio, and orientation varies along the z axis.

In component notation the magnetic field is

$$\begin{aligned} B_x &= -h[b(z)x + c(z)y] - \frac{k}{\kappa}[a'(z)x + b'(z)y], \\ B_y &= -h[a(z)x + b(z)y] - \frac{k}{\kappa}[b'(z)x + c'(z)y], \\ B_z &= \frac{k}{\kappa}[a(z) + c(z)], \end{aligned} \tag{8}$$

where $a'(z) = da(z)/dz$, etc. The contribution from H is a purely poloidal field which was derived as a solution of Eqs. (1) previously in [16]. The axial component of the field is contributed by the K term only, which is also already a solution of Eqs. (1). The fact that the superposition of both solutions is again a solution is nontrivial since Eqs. (1) are nonlinear.

In the limit $z \to \pm\infty$ the configuration becomes singular because the current density and the magnetic field grow without bounds. This squeezes the cross section of each pressure surface down to a line of finite length. For $\kappa z \gg 1$ the inclination $dy/dx = \sqrt{a_1/c_1}$ of the line is evident from $P^* \approx (a_1 + c_1)(\sqrt{a_1}\,x + \sqrt{c_1}\,y)^2 \exp 2\kappa z$. The maximal extension of the line is given by $x^2 \leq P^* c_1/(a_1 + c_1)$, $y^2 \leq P^* a_1/(a_1 + c_1)$. For $\kappa z \ll -1$ the situation is analogous, with the index 1 replaced by 2. Thus, the pressure surfaces consist of a middle region of finite thickness which gets progressively thinner towards both sides, with inclinations determined asymptotically by the coefficients a_1, a_2, c_1, c_2.

For the second type of solutions the functions H and K are given by

$$\begin{aligned} H(x,y,z) &= \tfrac{h\eta}{\kappa}[a(z)\cos\lambda x + c(z)\cos\lambda y], \\ K(x,y,z) &= \tfrac{k}{\lambda\kappa}[a(z)\cos\lambda x + c(z)\cos\lambda y], \end{aligned} \tag{9}$$

while the pressure P is determined by

$$\begin{aligned} P^*(x,y,z) \equiv & -2\frac{P - P_0}{(h^2 + k^2)\eta^2} = a^2(z)\sin^2\lambda x \\ &+ 2(1 - \cos\lambda x\cos\lambda y) \\ &+ c^2(z)\sin^2\lambda y. \end{aligned} \tag{10}$$

Here λ, with $\lambda^2 \leq \kappa^2$, is an arbitrary constant and η is defined by $\eta = (1 - \lambda^2/\kappa^2)^{1/2}$. Equations (5)–(7) are again valid, with the restriction $b_1 = b_2 = b(z) = 0$. A nontrivial solution is then possible for $a_2 = c_1 = 0$ only. With the normalization mentioned there results $a(z) = a_1 e^{\kappa z} = 1/c(z)$.

For $|\lambda x|, |\lambda y| \ll 1$, replacing $\sin\lambda x$, $\cos\lambda x$ by λx, $1 - (\lambda x)^2/2$, respectively, etc., the type-II pressure (10) takes on the same form as the type-I pressure (4) in the case $b(z) = 0$. Close to the axis, therefore, the surfaces also consist of ellipses, with a major axis in the x or y

direction. Farther away, with P^* increasing from its on-axis value zero, they are progressively deformed into a more rhomboidal shape.

In the limit $\kappa z \gg 1$, the pressure surfaces (10) approximately take on the form $c(z)P^*/4 = a(z)\sin^2(\lambda x/2) + c(z)\sin^2(\lambda y/2)$. This implies again that for $\kappa z \to \infty$ the pressure surfaces shrink down into a line, this time at $x = 0$ and with an extension $y^2 \le y_\infty^2$, where $\sin^2(\lambda y_\infty/2) = P^*/4$. For $\kappa z \ll -1$ the situation is analogous, with the roles of x and y interchanged.

The magnetic field of the type-II solution is

$$\begin{aligned} B_x &= -h\eta c(z)\sin\lambda y \; - \; k a(z)\sin\lambda x\,, \\ B_y &= -h\eta a(z)\sin\lambda x \; + \; k c(z)\sin\lambda y\,, \\ B_z &= k\lambda[a(z)\cos\lambda x \; + \; c(z)\cos\lambda y]/\kappa\,. \end{aligned} \tag{11}$$

Again, the contribution from H is a purely poloidal field. It was derived as a solution of Eqs. (1) in [14]. The axial component of the field is contributed by the K term only which, as an independent solution, was found in [17]. Again, these two solutions can be superposed to give a new solution of the nonlinear equilibrium problem.

The current density components for the type-II solutions are

$$\begin{aligned} j_x &= \eta\kappa[-ha(z)\sin\lambda x \; + \; k\eta c(z)\sin\lambda y], \\ j_y &= \eta\kappa[+hc(z)\sin\lambda y \; - \; k\eta a(z)\sin\lambda x], \\ j_z &= \eta\lambda h[a(z)\cos\lambda x \; + \; c(z)\cos\lambda y]. \end{aligned} \tag{12}$$

If the perpendicular and the axial scale lengths κ and λ coincide, η, and therefore the current density **j**, vanishes.

An example of a type-II solution is given in Fig. 1. It shows two nested pressure surfaces, with $P^* = 0.7$ (inner surface) and $P^* = 2.0$ (outer surface). Further parameters are $\kappa = \sqrt{2}$, $\lambda = 1$, and $a_1 = 1$. The axial region presented extends from $z = -1.4$ to $z = +1.4$. The figure might also serve for a qualitative representation of a type-I equilibrium (with asymptotic inclinations of 0° and 90°). In Fig. 2, which is a (enlarged) continuation of the outer surface of Fig. 1 to the left, and extends from $z = -4.4$ to $z = -1.4$, the asymptotic behavior for $z \to -\infty$ is rather evident.

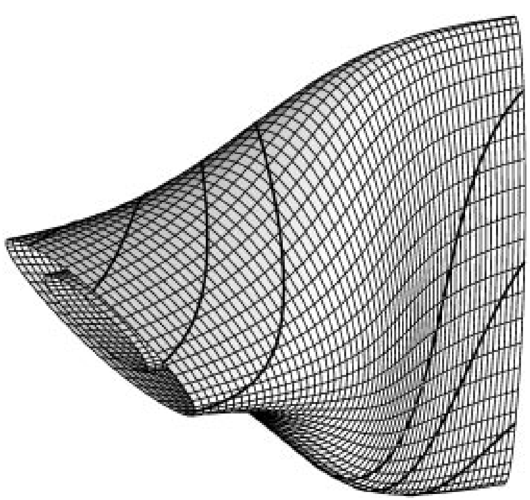

FIG. 1. Nested pressure surfaces of type-II MHD equilibria. Field lines rotate around the axis.

In Figs. 1 and 2 several field lines are also shown as thick lines. $h = 5, k = 1$ is used for these orbits. In polar coordinates r, θ, defined by $x = r\cos\theta, y = r\sin\theta$, the field line equations are

$$\frac{d\theta}{dz} = \frac{1}{rB_z}(B_y\cos\theta \; - \; B_x\sin\theta) \tag{13}$$

and an analogous equation for dr/dz.

The asymptotic behavior of the field lines can also be discussed analytically. In the limit $\kappa z \gg 1$ the field line twist turns out to vanish. All field lines eventually end up at a height y which depends on the starting point but does not change with z anymore. Together with $x = 0$ this implies $\theta = \pm\pi/2, \mathrm{modulo}\,\pi$. The analogous behavior results for $z \to -\infty$. In Fig. 2 this asymptotic behavior of the field lines is also evident.

In toroidal geometry the rotational transform ι, for unclosed field lines, is defined as $\iota = \lim_{\phi\to\infty} M(\phi)/N(\phi)$, where ϕ is some toroidal coordinate and $M(\phi), N(\phi)$ are the number of poloidal and toroidal turns, respectively, of a field line around the axis. This limit is equivalent to averaging over the total surface, and in consequence ι and the shear $s = d\iota/dV$ are constant on pressure surfaces (V is a flux label of the surface, e.g., the pressure itself). Since the present equilibria, however, do not have an axial periodicity this averaging property does not take place, and ι differs, in principle, from field line to field line.

A natural definition of the rotational transform $\iota(x_0)$ in the present case would seem to be the limit of the number of turns from $z = -\infty$ to $+\infty$ as a function of the starting coordinate x_0, with $N = 1$. According to the discussion of $\theta(z)$ above, however, $\iota(x_0)$ then takes on only half-integer values, $\iota(x_0) = \pm m/2$, with at most two consecutive integer values m, for a given equilibrium.

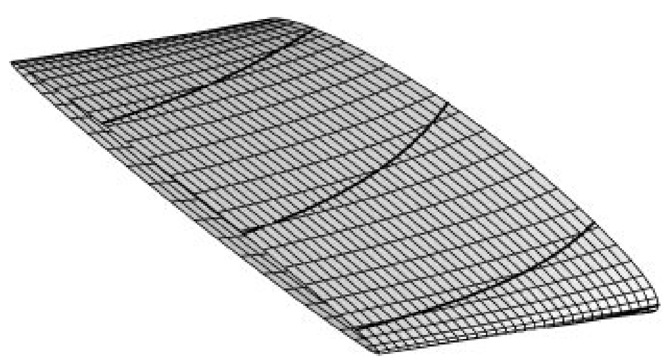

FIG. 2. Continuation of the outer surface of Fig. 1 to the left. Asymptotically, the surface becomes squeezed flat and the field lines approach a constant distance to the axis.

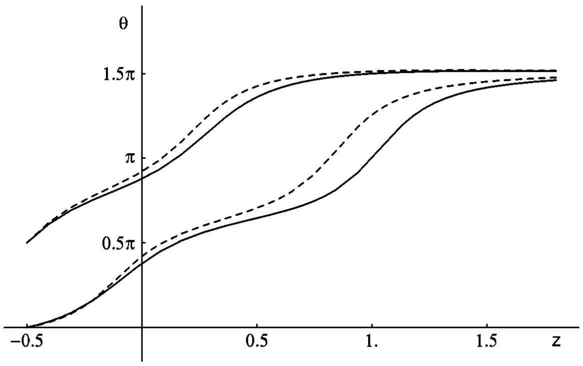

FIG. 3. Poloidal angle θ of field lines for low pressure, $P^* = 0.5$ (solid lines), and high pressure $P^* = 2$ (dashed lines). Starting values are $\theta_0 = 0$ (lower pair) and $\theta_0 = \pi/2$ (upper pair). Difference between solid and dashed curves indicates field line shear.

A reasonable definition of the shear with this ι is not possible.

Global definitions of ι, thus, do not seem appropriate here. In many respects, however, *local* values of the field line slope $\theta' = d\theta/dz$ are also relevant anyway. θ' not only varies along z and from field line to field line but, for type-II solutions, also from pressure surface to pressure surface. This is evident from Fig. 3 where the angle $\theta(z)$ is plotted for two field lines for a type-II solution in the region, $z \in [-0.5, 1.8]$. The field lines start at $\theta = 0$ and $\theta = \pi/2$, respectively. (These two values are selected for ease of identification.) The solid curves correspond to $P^* = 0.5$ and the dashed curves to $P^* = 2.0$, with h, k, κ, λ, and a_1 as in Figs. 1 and 2. The difference between the solid and the dashed curves confirms the aforementioned existence of a "radial" derivative of $d\theta/dz$ which could serve as a local substitute for the shear.

In type-I solutions the shear is absent: B_x, B_y are linear in r, while B_z is independent of r. In consequence, the variable r completely drops out of Eq. (13). The field line trajectories on different pressure surfaces are all equivalent.

For all toroidal MHD equilibria with closed field lines the so-called current-closure condition [3] for $F := \oint dl/|\mathbf{B}|$ has to hold, namely, that F be the same for all closed field lines on a fixed pressure surface. Although the present equilibria are not topologically toroidal and the field lines are not closed, except for $k = 0$, it is nevertheless found that F is constant on $P =$ const if the integration along the unclosed field lines is extended from $z = -\infty$ and $+\infty$. For type-I solutions it is found that $F = \pi[(a_1 + c_1)(a_2 + c_2)]^{-0.5}/(2|k|)$ for $k \neq 0$, and $F = 2\pi/|h|$ for $k = 0$, i.e., F does not even depend on P at all. For type-II solutions there results $F = \mathbf{K}(\sqrt{P^*/4})/(|k\lambda|)$, for $k \neq 0$, and $F = 4\mathbf{K}(\sqrt{P^*/4})/(|h\lambda\eta|)$, for $k = 0$, where $\mathbf{K}$ is the complete elliptic integral of the first kind.

The equilibria just described have, of course, the serious drawback that toroidal curvature is missing and that they are not even periodic in the axial direction. An answer to the question whether *periodic* 3D equilibria with straight axis and twisted field lines exist or not would be especially interesting since the potential for island formation and ergodization of field lines is then the same as in toroidal geometry, but complications from toroidal curvature are still absent.

[1] H. K. Moffatt, J. Fluid Mech. **159**, 359 (1985).
[2] A. I. Morozov and L. S. Solovev, in *Reviews of Plasma Physics,* edited by M. A. Leontovich (Consultants Bureau, New York, 1966), Vol. 2.
[3] L. S. Solovev and V. D. Shafranov, in *Reviews of Plasma Physics,* edited by M. A. Leontovich (Consultants Bureau, New York, 1970), Vol. 5, p. 1.
[4] J. Edenstrasser, J. Plasma Phys. **24**, 299 (1980).
[5] R. Lüst and A. Schlüter, Z. Naturforsch. **12a**, 850 (1957); H. Grad and H. Rubin, in *Proceedings of the Second United Nations Conference on the Peaceful Uses of Atomic Energy, Geneva, 1958,* edited by United Nations (United Nations Publication, Geneva, 1958), Vol. 31, p. 190; V. D. Shafranov, Zh. Eksp. Teor. Fiz. **33**, 710 (1957) [Sov. Phys. JETP **6**, 545 (1958)].
[6] D. Lortz, Z. Angew. Math. Phys. ZAMP **21**, 196 (1970).
[7] L. Spitzer, Phys. Fluids **1**, 253 (1958).
[8] T. Hayashi, T. Sato, and A. Takei, Phys. Fluids B **2**, 329 (1990).
[9] H. Grad, Phys. Fluids **10**, 137 (1967); Int. J. Fusion Energy **3**, 33 (1985).
[10] R. Kaiser, A. Salat, and J. Tataronis, Phys. Plasmas **2**, 1599 (1995).
[11] A. Salat, Phys. Plasmas **2**, 1652 (1995).
[12] D. Palumbo, Proc. Acad. Sci. Arts Palermo, Ser. 5, Pt. 1: **Sci IV**, 475 (1984).
[13] D. A. Garren and A. H. Boozer, Phys. Fluids B **3**, 2822 (1991).
[14] M. L. Woolley, J. Plasma Phys. **17**, 259 (1977).
[15] B. K. Shivamoggi, Quarterly Appl. Math. **XLIV**, 487 (1986).
[16] A. Salat and R. Kaiser, Phys. Plasmas **2**, 3777 (1995).
[17] R. Kaiser and A. Salat, J. Plasma Phys. (to be published).