Algebraic Tails of Probability Density Functions in the Random-Force-Driven Burgers Turbulence

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The dynamics of velocity fluctuations governed by the Burgers equation, driven by the white-in-time random forcing function with $\overline{[f(x+r,t)-f(x,t')]^2} \propto r^{\xi}\delta(t-t')$ is considered on the interval $0<x<L$. The properties of the probability density function of velocity differences $P(\Delta u, r)$ are investigated for the three cases $\xi = \{0; 1/2; 2\}$. It is shown that the tail of the probability density function in the interval $\Delta u/r^z \ll -1$; $|\Delta u| \ll u_{\rm rms}$ and $r \ll L$ is accurately described by the asymptotic algebraic relation $\mathcal{P}(\Delta u, r) \propto r/(\Delta u)^{\gamma}$ with $\gamma = 1 + 1/z$, where $z = (\xi+1)/3$. A detailed numerical investigation, performed in this work, supports this result. [S0031-9007(96)01384-1]

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Recent interest in the problem of turbulence in a random-force-driven Burgers equation is motivated by the possibility that this system might play a role in the development of turbulence theory similar to that played by the two-dimensional Ising model in creation of the theory of critical phenomena. It was demonstrated recently [1] that the statistical properties of solution of the one-dimensional Burgers equation, driven by a white-in-time random force with the correlation function $\overline{f(k,t)\,(k',t')} \propto k^{-1}\delta(t-t')$, have many features surprisingly similar to the ones observed in the real-life three-dimensional turbulence: Kolmogorov energy spectrum, intermittency and the scaling properties of the dissipation rate fluctuations, and bifractality of the probability density function (PDF) of velocity differences $\mathcal{P}(\Delta u, r)$, where $\Delta u \equiv u(x+r) - u(x)$. It was shown that intermittency in this system is a consequence of the algebraic tail of the probability density function for $\Delta u < 0$, allowing scaling solutions $S_p = \overline{(\Delta u)^p} \propto r^{p/3}$ only for the moments with $p<3$ while the moments with $p>3$ scale as $S_p \propto r u_{\rm rms}^{p-3}$ and depend on the single-point property $u_{\rm rms} \equiv \sqrt{\overline{u^2}}$. The shape of the algebraic tail of the PDF $P(\Delta u, r) \propto r/(\Delta u)^4$ was evaluated from the balance equations with the assumption that the dissipation in the system is dominated by the well separated shocks.

The groundbreaking theoretical work by Polyakov [2] dealt with the same problem, but driven by the force with the correlation function concentrated at the largest scales only: $D(r) = \overline{f(x)f(x+r)} \propto \kappa(0) - ar^2$, when $r/L \ll 1$, and where L is the length scale of the forcing function, so that $D(k) = 0$ when $k \gg 2\pi/L$. Polyakov produced an exact solution based on some self-consistent assumptions about the structure of the theory. The common feature of the solutions presented in [1,2] is the existence of the algebraic tails of the probability density function of velocity difference with the exponents depending on the properties of the forcing function. The algebraic tails of $\mathcal{P}(\Delta u, r)$ in Polyakov's problem have recently been obtained in the work by Sinai *et al.* [3] using a different method. The goal of the present Letter is to investigate the universal features of a random-force-driven Burgers equation considering few different cases of the stirring force. It will be shown that the main physical conclusions of the work [2] are valid disregarding the power spectrum of the driving force. However, the exponent of the algebraic tail of the probability density in the limit $\Delta u/r \ll -1$ with $|\Delta u| \ll u_{\rm rms}$ and $r \ll L$, dominated by the well separated shocks, is represented by a general expression similar to the one derived in [1].

Let us consider the one-dimensional problem

$$u_t + uu_x = f + \nu u_{xx}\,, \tag{1}$$

where the white-in-time random forcing function is defined by its variance and $\overline{f(x,t)f(x',t')} = \kappa(r)\delta(t-t')$, where $r = |x - x'|$ and when $r/L \ll 1$ the velocity structure function is $\kappa(r) = \kappa(0) - ar^{\xi}$. According to the recent theoretical and numerical works [1,2], the probability density of the velocity differences $\mathcal{P}(\Delta u, r)$ consists of two physically different regions. When $|\Delta u| > u_{\rm rms}$ the PDF

$$(\Delta u, r) = r\Theta\left(\frac{\Delta u}{u_{\rm rms}}\right) \tag{2}$$

depends on the single-point property $u_{\rm rms}$ and, therefore, is not a universal function. At the same time, in the interval $|\Delta u| \ll u_{\rm rms}$ and $r \ll L$ the PDF can be represented in the universal scaling form

$$\mathcal{P}(\Delta u, r) = \frac{1}{r^z}F\left(\frac{\Delta u}{r^z}\right), \tag{3}$$

where $\int_{-\infty}^{+\infty} F(x)\,dx = 1$. The dynamic exponent z can be calculated using the equation similar to that derived by Polyakov: $z = (\xi+1)/3$. For $x \equiv \Delta u/r^z \gg 1$, the universal scaling function in (3) is given by the expression $F(x) \propto \exp[-\alpha(z)x^3]$ where the coefficient $\alpha(z)$ can be evaluated from the theory. In the region $|x| \ll 1$ the PDF

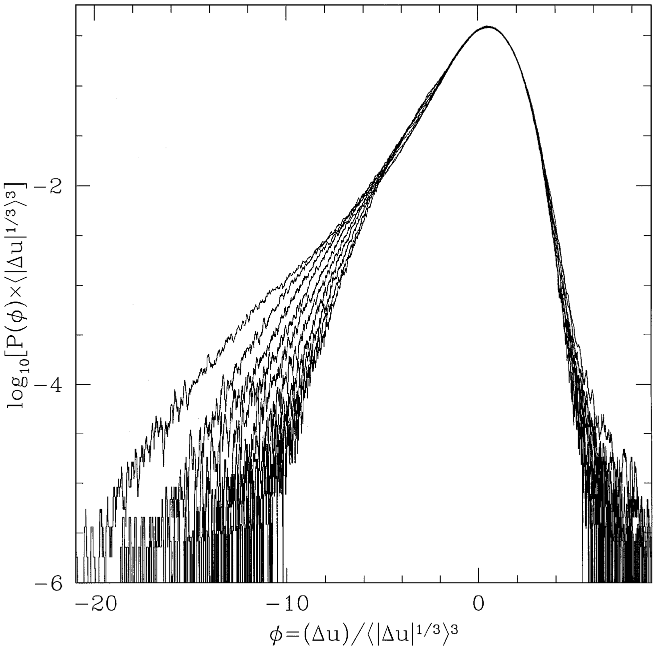

FIG. 1. Collapse of the PDFs in the universal region of Δu for the $z = 1/3$ case.

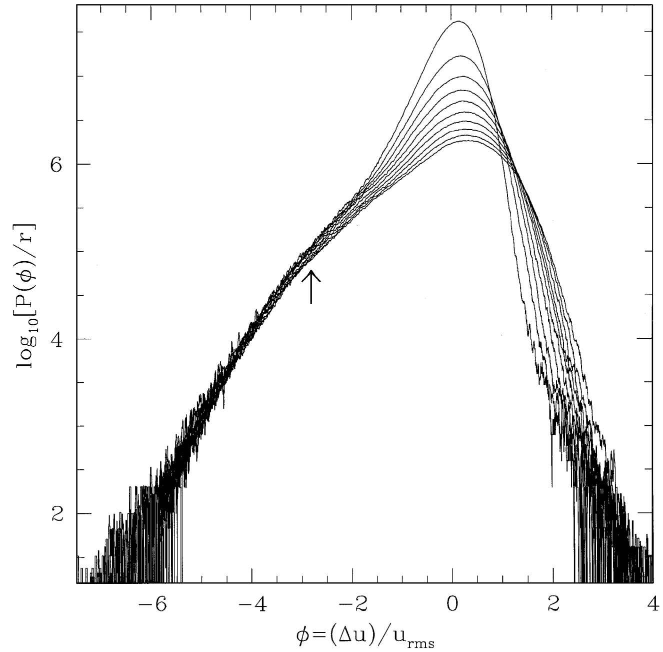

FIG. 2. Collapse of the PDFs in the nonuniversal region of Δu for the $z = 1/3$ case.

is a solution of the Fokker-Plank-like equation [2]. In the most interesting interval $|\Delta u| \ll u_{\rm rms}$, $\Delta u < 0$ and $|x| \gg 1$ the probability density is dominated by the well separated shocks and that is why it can be represented in a general form [1]: $\mathcal{P}(\Delta u, r) \approx (r/L)\mathcal{P}(U)$, where r/L is a probability to find a shock in the interval of the length r, and $\mathcal{P}(U)$ is the probability of a shock having the amplitude U. Since in this range the shocks are well separated, $\Delta u \approx U$. If the PDF has algebraic tails, i.e.,

$$\mathcal{P}(\Delta u, r) \propto \frac{r}{(\Delta u)^\gamma}, \tag{4}$$

then the exponent γ can be evaluated by equating the above expressions with the only algebraic solution with

$$\gamma = 1 + \frac{1}{z}. \tag{5}$$

This result has been tested on three different cases $z = \{1/3; 1/2; 1\}$. Case $z = 1/3$ corresponds to the logarithmic flux case considered in [1], case $z = 1$ is the large-scale forced case considered in [4–3], and $z = 1/2$ is an intermediate case between them. The numerical part was described in [1]; therefore we will mostly dwell here on the discussion of the results. For all of the cases $z = \{1/3, 1/2, 1\}$ the estimate for the dissipative cutoff wave number is $k_d \approx 600$. Measurements of the PDFs were made for the following values of $r \in \{100, 200, 300, \ldots, 1000\}dx$, where $dx = 2\pi/12\,288$ is the distance between spatial grid points. We can see that the viscous effects can be safely neglected for the values of displacement $r > 20dx$. At the same time it is dangerous to take r too large as well, because the asymptotic relations (4) and (5) are valid only for $r/L \ll 1$. First, we direct our attention to the general properties of the PDF, predicted by (2) and (3), for each of the considered values of z. We measure $\mathcal{P}(\Delta u, r)$ corresponding to various separations r in the universal range ($k_d \ll r \ll L$, where k_d is an ultraviolet dissipative cut-

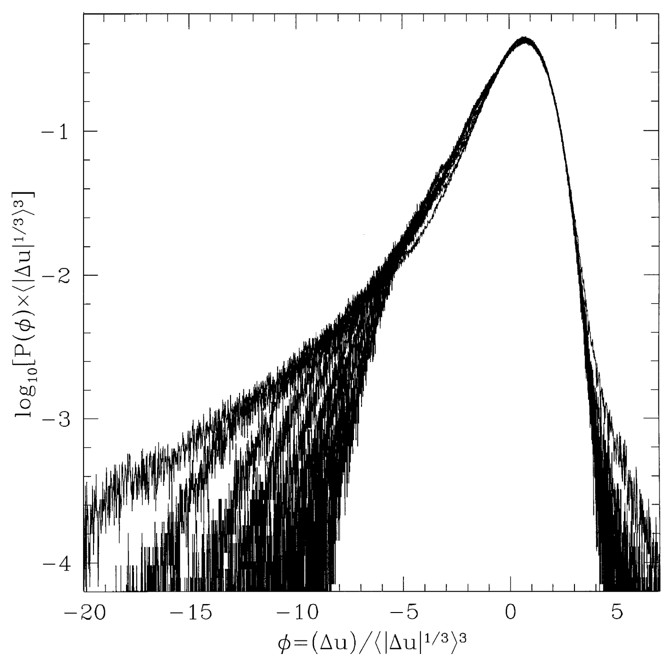

FIG. 3. Collapse of the PDFs in the universal region of Δu for the $z = 1/2$ case.

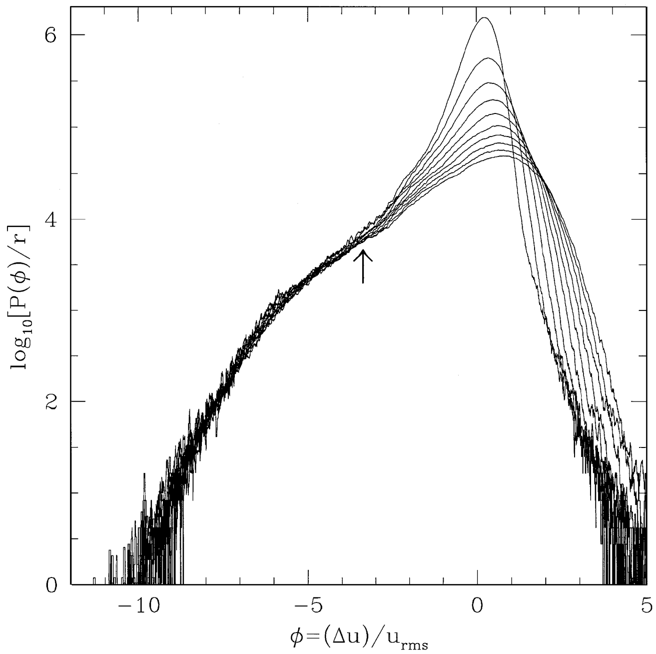

FIG. 4. Collapse of the PDFs in the nonuniversal region of Δu for the $z = 1/2$ case.

off), and produce scaling functions $\Theta(x)$ and $F(x)$. If (2) and (3) do hold true, then the curves for various values of r should approximately collapse in the appropriate regions of Δu and r discussed above. This is indeed observed in Figs. 1–6 where we have plotted collapsed PDFs in universal and nonuniversal regions defined by (3) and (2), respectively. The results presented in Figs. 1–6 confirm

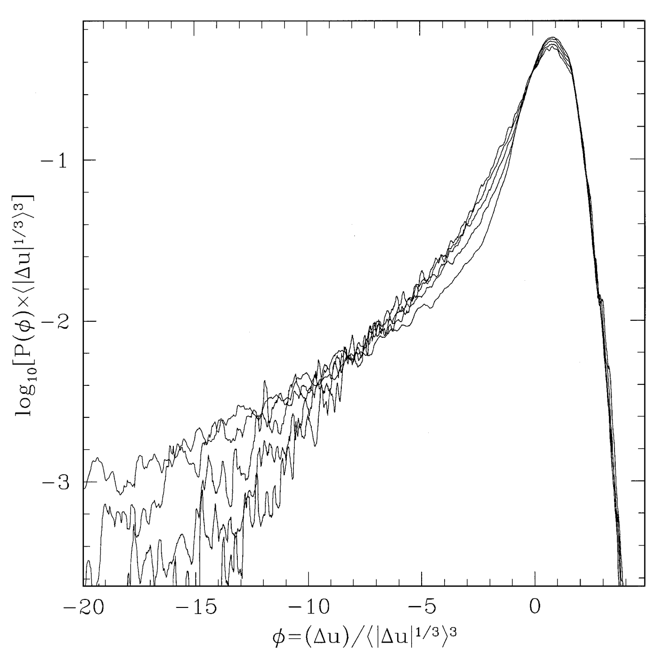

FIG. 5. Collapse of the PDFs in the nonuniversal region of Δu for the $z = 1$ case.

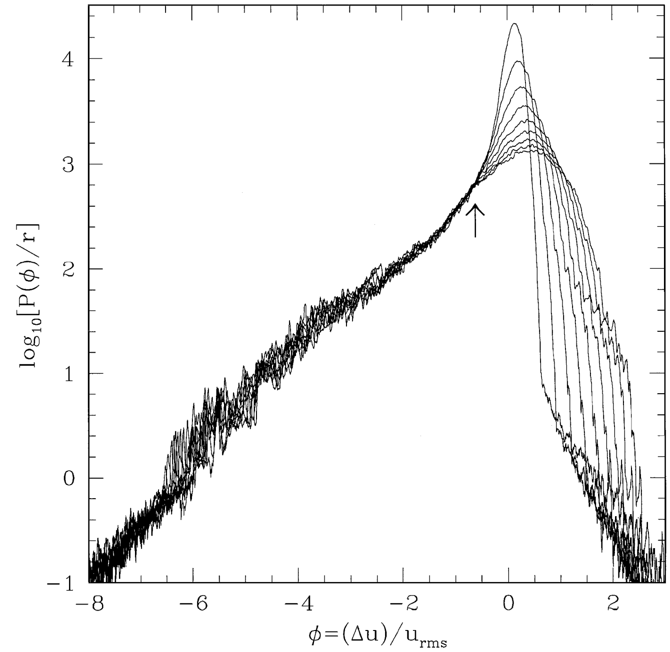

FIG. 6. Collapse of the PDFs in the nonuniversal region of Δu for the $z = 1$ case.

the main conclusions of the Polyakov theory [2], which in a general case are reformulated as following: when $x = \Delta u/r^z > 0$, $|x| \leq O(1)$, and $\Delta u \ll u_{\rm rms}$, the PDF is given by the scaling relations (2) and (3). However, on each of the curves we can detect a small bump around $x = \Delta u/(\Delta u)_{\rm rms} \approx -1$, corresponding to a crossover to the shock-dominated regime ($x \ll -1$ and $|\Delta u| \ll u_{\rm rms}$),

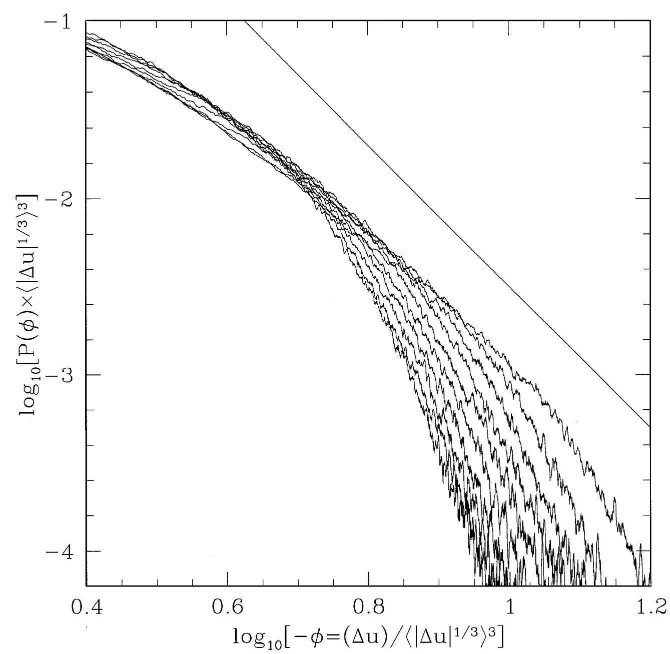

FIG. 7. Log-log plot of the negative Δu tail for the $z = 1/3$ case.

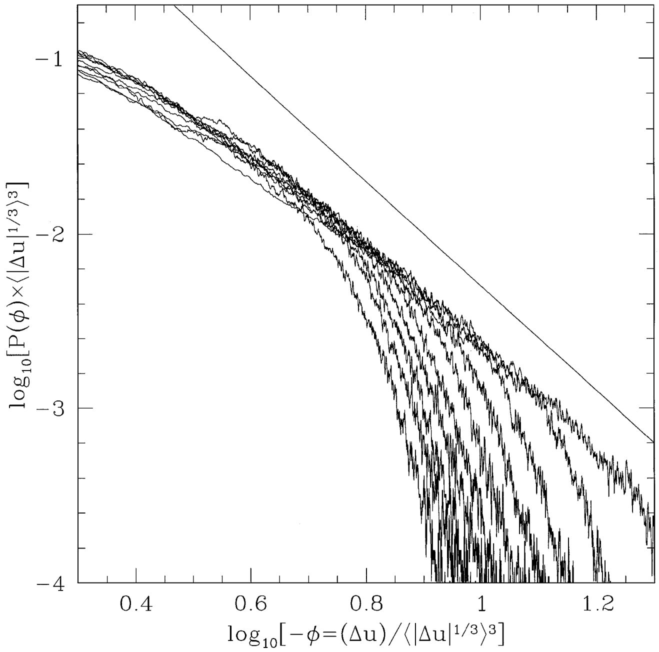

FIG. 8. Log-log plot of the negative Δu tail for the $z = 1/2$ case.

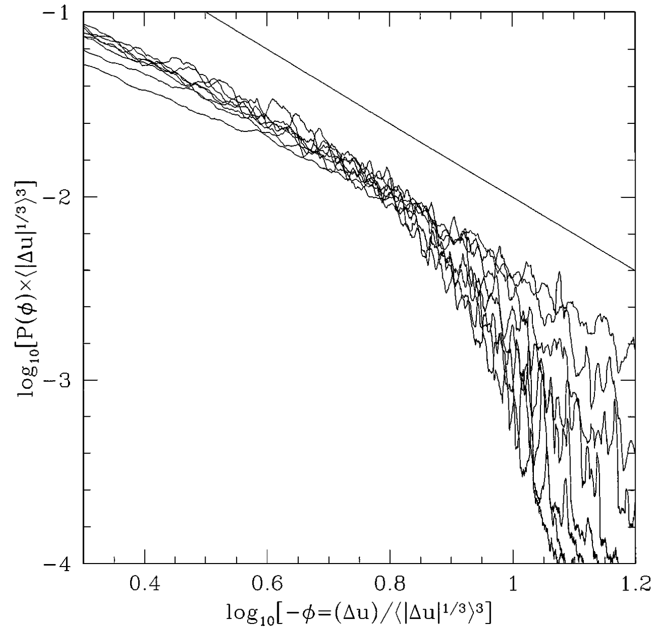

FIG. 9. Log-log plot of the negative Δu tail for the $z = 1$ case.

characterized by expressions (4) and (5), consistent with the outcome of [1]. The fact that in this range the quality of the collapse is a little bit less good than that in the interval $|x| < 1$ reflects the fact that it is more difficult to reduce statistical error in the process dominated by the well pronounced single shocks. We also see that plotted in the coordinate $\Delta u/u_{\text{rms}}$, all functions collapse starting from the point $\Delta u/u_{\text{rms}} \approx 1$, in accord with the Polyakov general description.

Now we discuss the quantitative test of prediction (4) and (5). We are dealing here with the system having a finite number of degrees of freedom (particularly, 12 288 Fourier modes in spectral representation). Thus, investigating the asymptotic relation (4), valid in the interval $x \ll -1$ and $|\Delta u| \ll u_{\text{rms}}$ may be quite difficult due to the limited range of the values of displacement r available to us: to make this range as wide as desired, we have to be able to consider $r/L \to 0$ which is not an easy task due to the hardware limitations. In Figs. 7–9 the comparison of the relation (4) (straight line in a log-log scale) with the calculated PDFs, is presented. As one may observe, the relations (4) and (5) are confirmed with a good accuracy. It is worth mentioning that even closer agreement between theory and numerics is found when the observed probability density of the shock amplitudes $P(U)$ is compared with the theoretical predictions [1].

The algebraic tails of the PDF of velocity differences described in this Letter may turn out to be quite general. It is very interesting to understand how the results, presented here, depend on the space dimensionality d. Our own numerical studies of the random-force-driven three-dimensional Burgers equation [5] show results similar to the ones presented here. This will be the subject of a future communication.

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