

## Mesoscopic Capacitors: A Statistical Analysis

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The capacitance of mesoscopic samples depends on their geometry *and* physical properties, described in terms of characteristic times scales. The resulting ac admittance shows sample to sample fluctuations. Their distribution is studied here—through a random-matrix model—for a chaotic cavity capacitively coupled to a backgate: it is observed from the distribution of scattering time delays for the cavity, which is found analytically for the orthogonal, unitary, and symplectic universality classes, one mode in the lead connecting the cavity to the reservoir and no direct scattering. The results agree with numerical simulations. [S0031-9007(96)01262-8]

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The elementary notion of capacitance of a system of conductors, as a quantity determined solely by the geometry, has to be revised if the electric field is not completely screened at the surface of the conductors. In fact, the penetration distance of the field is of the order of the Thomas-Fermi screening length, which may be appreciable for a mesoscopic conductor: the standard description of a capacitor in terms of the *geometric capacitance*  $C_e$  (that relates the charge  $Q$  on the plate to the voltage  $U$  across the capacitor) gives way, in the mesoscopic domain, to a more complex entity  $C_\mu$ , the *electrochemical capacitance* (that relates  $Q$  to the electrochemical potential of the reservoirs), which depends on the properties of the conductors [1]. This fact, in turn, has important consequences for the ac current induced in the system when the electrochemical potentials are subject to a nonzero-frequency time variation [1].

The electrochemical nature of the capacitance has been relevant to a number of experiments [2] and has been discussed theoretically by several authors [1,3,4]. Remarkably, it has been found that the resulting ac admittance can be described in terms of characteristic time scales related to energy derivatives of scattering matrix elements.

It is well known that, as a result of quantum interference, the dc conductance of mesoscopic structures shows strong fluctuations as a function of the Fermi energy or the magnetic field, as well as from sample to sample. A statistical analysis of this phenomenon has been done, for diffusive transport in disordered structures, using microscopic perturbative and macroscopic random-matrix theories [5], and for ballistic microstructures—cavities in which impurity scattering can be neglected so that only scattering from the boundaries is important—whose classical dynamics is chaotic, using semiclassical, field theoretic and random-matrix approaches [6,7].

An extension of the above random-matrix studies to include the ac admittance of mesoscopic structures is the subject of the present investigation.

In this Letter we shall confine our discussion to the geometry shown in Fig. 1. In this system there is, of course, no dc transport, but there may be an ac current, determined by the admittance [1,3]

$$g^I(\omega) = \frac{g(\omega)}{1 + (i/\omega C_e)g(\omega)} \equiv -i\omega C_\mu + \dots, \quad (1)$$

written in the Thomas-Fermi approximation and to lowest order in the frequency  $\omega$ . Here,  $g(\omega)$ ,  $g^I(\omega)$  denote the admittance for the noninteracting and interacting system, respectively, the former being given, for zero temperature, by

$$g(\omega) = -i\omega e^2 \left\{ \frac{1}{2\pi i} \text{Tr} \left[ S^\dagger(E) \frac{\partial S(E)}{\partial E} \right] \right\} + \dots \\ = -i\omega e^2 N\tau/\Delta + \dots \quad (2)$$

Here,  $S(E)$  is the  $N \times N$  scattering matrix for the system formed by the cavity and the lead,  $N$  being the number of propagating modes, or open channels, in the lead;  $\Delta$  is the mean level spacing for the cavity (the inverse of the level density). Following [8], we have introduced the *dimensionless time delay*

$$\tau = \frac{\Delta}{2\pi N} \frac{\partial \theta}{\partial E}, \quad (3)$$

where  $\exp(i\theta) = \det S$ . We then write  $g^I(\omega)$  of Eq. (1) as

$$g^I(\omega) = -i\omega C_e \alpha + \dots, \quad (4)$$

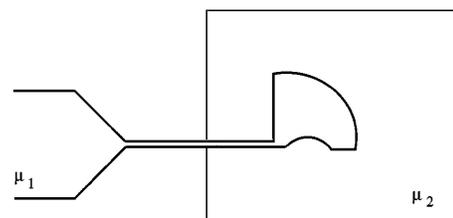


FIG. 1. Mesoscopic capacitor: A cavity (thick line) is connected via a perfect lead to reservoir 1 and capacitively coupled to a macroscopic back gate (thin line) connected to reservoir 2. The cavity is ballistic, and its classical dynamics is chaotic.

where the dimensionless capacitance  $\alpha$  is given by

$$\alpha = C_\mu/C_e = \frac{\tau}{\tau + \eta} \tag{5}$$

and

$$\eta = \frac{C_e}{N(e^2/\Delta)}. \tag{6}$$

Notice that, for a macroscopic cavity,  $\eta \ll 1$ , so that  $\alpha \approx 1$  and  $g^l(\omega) \approx -i\omega C_e$ .

The *one-energy* statistical distribution of the  $S$  matrix for ballistic cavities larger than the Fermi wavelength has been modeled successfully through an “equal *a priori* probability” ansatz (known as a “circular ensemble”) [6,7], when the classical dynamics is chaotic and direct processes though the microstructure can be neglected, so that, as a result, the averaged  $S$  vanishes,  $\bar{S} = 0$ . It is clear, through, that the time delay  $\tau$  of Eq. (3) is a *two-energy* function and thus requires more information for its statistical study. The distribution of  $\tau$ ,  $w(\tau)$ , has been studied for a one-dimensional disordered system within the invariant imbedding formalism in [9]. In another approach, an underlying Hamiltonian described by a Gaussian ensemble was assumed and the problem analyzed using supersymmetry techniques: the two-point correlation function for the  $S$  matrix elements was derived in [10]; phase-shift times for unitary symmetry,  $N$  and  $\bar{S}$  arbitrary were studied in [11]. Reference [12] finds an approximation to  $w(\tau)$ . We concentrate, in what follows, on  $w(\tau)$  for arbitrary symmetry (orthogonal, unitary, and symplectic, identified as  $\beta = 1, 2$ , and  $4$ , respectively),  $N = 1$  and  $\bar{S} = 0$ : we show that this case can be treated using an old conjecture by Wigner [13,14]. We believe that the simplicity of the argument is appealing and gives an interesting perspective to the problem and a unified point of view for arbitrary  $\beta$ . We also remark that, for ballistic cavities, the case of just one open channel,  $N = 1$ , is very relevant from an experimental point of view, since cases of small  $N$  have been realized in the laboratory [15]. We find below

$$w_\beta(\tau) = \frac{(\beta/2)^{\beta/2}}{\Gamma(\beta/2)} \frac{e^{-\beta/2\tau}}{\tau^{(\beta+4)/2}}, \tag{7}$$

where  $0 \leq \tau < \infty$ . For  $\beta = 2$ , this result agrees with that of Ref. [11]. The main result of the present paper, i.e., the  $\beta$  dependent distribution of the dimensionless capacitance  $\alpha$  [ $\alpha$  is related to the ac admittance via Eq. (4)], then follows as

$$p_{\beta,\eta}(\alpha) = \frac{(\beta/2)^{\beta/2}}{\Gamma(\beta/2)} \frac{(1 - \alpha)^{\beta/2}}{\eta^{(\beta+2)/2} \alpha^{(\beta+4)/2}} e^{-\beta(1-\alpha)/2\eta\alpha}, \tag{8}$$

for  $0 \leq \alpha \leq 1$ . A plot of  $p_{1,\eta}(\alpha)$  for various values of  $\eta$  is presented in Fig. 2. For a macroscopic cavity,  $\eta \rightarrow 0$  and  $p_{\beta,\eta}(\alpha) \rightarrow \delta(1 - \alpha)$ . We now derive the distribution of time delays, Eq. (7).

We write  $S$  for  $N = 1$  as

$$S(E) = \frac{1 + iK(E)}{1 - iK(E)} = e^{i\theta(E)}. \tag{9}$$

For pure resonance scattering the  $K$  function can be given the sum-over-resonance form [13,14]

$$K(E) = \sum_\lambda \frac{\Gamma_\lambda}{E_\lambda - E}, \tag{10}$$

where the “widths”  $\Gamma_\lambda$  for a given symmetry class  $\beta$  can be written in terms of real amplitudes  $\gamma_\lambda^{(i)}$  as

$$\Gamma_\lambda = \sum_{i=1}^\beta [\gamma_\lambda^{(i)}]^2. \tag{11}$$

The quantity  $\theta'(E)/2 = h(E)$  was studied extensively by Wigner [13,14]; it is called the “invariant derivative,” because it remains invariant under the transformation

$$K_\phi = \frac{K + \tan \phi}{1 - K \tan \phi}, \tag{12}$$

$\phi$  being constant; since  $K = \tan(\theta/2)$ , (12) takes  $\theta/2$  to  $\theta/2 + \phi$ , and hence  $S$  to  $e^{i\phi} S e^{i\phi}$ . Both  $K$  and its transforms have the form  $K = \tan \int_c^E h(E) dE$ ,  $c$  being different for different transforms. Starting from one pole  $E_1$  of  $K$ , one can obtain the next one by determining the abscissa  $E_2$  so that the area under  $h(E)$  between  $E_1$  and  $E_2$  is  $\pi$ . Moreover, at a pole  $E_\lambda$  we have  $\Gamma_\lambda = 1/h(E_\lambda)$ . These relations are shown in Fig. 3. The levels and widths of the transforms of  $K$  can be obtained by a similar construction, starting at another abscissa.

From (9) and (10) we find the energy average of  $S(E)$  as

$$\overline{S(E)} = S(E + iI) = \frac{1 - t}{1 + t}, \tag{13}$$

where  $I \rightarrow \infty$  [16] and  $t = \pi \bar{\Gamma}/\Delta$ . For  $\overline{S(E)} = 0$  [circular ensemble, invariant under (12)], we have  $t = 1$ . In this case (referred to in Refs. [13,14] as that of a “normalized”  $R$  function) Wigner proposes the conjecture: *the*

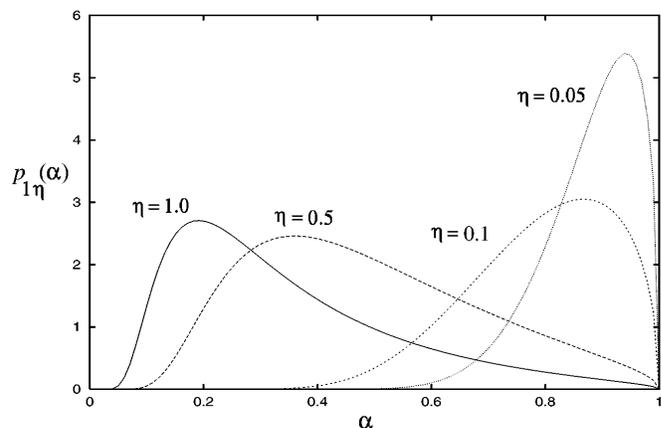


FIG. 2. The probability density of  $\alpha$ —the ratio of the electrochemical to the geometric capacitance—for the orthogonal case [Eq. (8)], for a number of values of  $\eta$ .

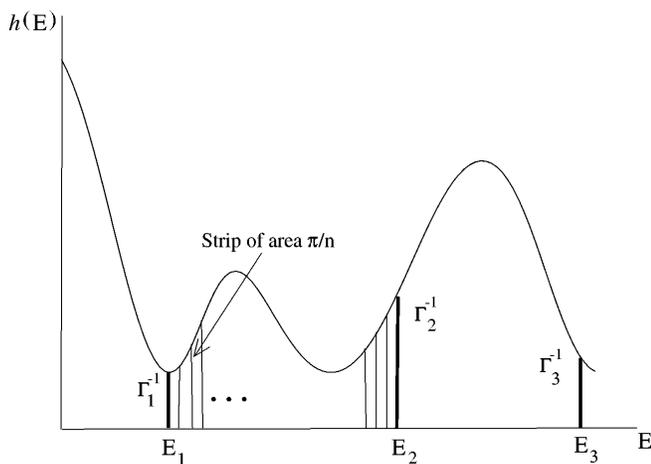


FIG. 3. The invariant derivative  $h(E)$ . The  $E_\lambda$  are the poles of  $K(E)$  and the  $\Gamma_\lambda$  the corresponding widths. Replicas of  $K$  with the same width distribution (according to Wigner's conjecture) are generated via the transformation (12) and used to subdivide the area between successive levels into  $n$  strips.

*statistical distributions of level spacings and residues are invariant under the transformation (12).*

The above statement is a “conjecture,” not a “theorem,” and it is not clear, *a priori*, for what distributions, if any, it is fulfilled. Wigner, in his papers, proposes it for “most statistical distributions.” The conjecture, in relation with the residue distribution, was verified numerically for the case in which the energy levels entering Eq. (10) are constructed from a Gaussian orthogonal, unitary, or symplectic ensemble, and the  $\gamma_\lambda^{(i)}$  of Eq. (11) as independent Gaussian variables: the residue distribution was found to remain invariant, within the statistical error bars of the numerical simulation. On the other hand, the conjecture is seen, in our numerical studies, to be violated for a spectrum of statistically independent energy levels following a Poisson distribution.

Call  $Q(h)$  the probability density of the inverse widths  $h(E_\lambda) = h_\lambda$  and  $P(h)$  the probability density of  $h$  across the energy axis, irrespective of whether we are at resonance or not;  $P(h)$  is related to  $w(\tau)$  as  $w(\tau) = (\pi/\Delta)P(\pi\tau/\Delta)$ . Assuming the above conjecture, Ref. [14] shows that

$$P(h) = \frac{\pi}{h\Delta} Q(h). \quad (14)$$

This relation can be understood by means of a very simple argument. Consider, for one given  $K$ , the following *level-average*:

$$\langle f(h) \rangle_\lambda = \frac{1}{m} \sum_{\lambda=1}^m f(h_\lambda), \quad (15)$$

for an arbitrary function  $f$ . From Fig. 3 we see that we cannot replace the sum in this equation by an integral. However, using the transformation (12) we can construct

“replicas” of  $K$ , all having the same distribution of  $h_\lambda$ ; we do this  $n$  times, in such a way that the area between two successive levels is subdivided into  $n$  strips of area  $\pi/n$  each. Now we have a fine mesh, the sum over which can be approximated by an integral, using a density  $nh/\pi$ , since the base of one of the above strips, at the place where  $h$  is the local value of the curve, is  $\pi/nh$ . We then arrive at the above relation (14).

If we use, in (14), the variable  $u = \pi/h\Delta$  (and denote the distributions with a hat), we have

$$\hat{P}(u) = u\hat{Q}(u). \quad (16)$$

On the left-hand side (LHS)  $u$  can be thought of in terms of  $\tau$  of Eq. (3) as  $u = 1/\tau$ ; at resonance,  $u$  takes the value  $u_\lambda = \pi\Gamma_\lambda/\Delta$ , which is the relevant variable on the right-hand side (RHS) of Eq. (16). Thus, knowing the distribution of widths  $\hat{Q}(u)$ , Eq. (16) allows finding  $\hat{P}(u)$  [17].

For the three universality classes  $\beta = 1, 2, 4$  and in dependent Gaussian variables  $\gamma_\lambda^{(i)}$ , the distribution  $\hat{Q}(u)$  is the  $\chi^2$  distribution function with  $\beta$  degrees of freedom,

$$\hat{Q}_\beta(u) = \frac{(\beta/2)^{\beta/2}}{\Gamma(\beta/2)} u^{(\beta-2)/2} e^{-(\beta/2)u}; \quad (17)$$

Eq. (16) then gives  $\hat{P}(u)$ , from which we find the distribution of time delays  $w_\beta(\tau)$  of Eq. (7). We notice the remarkable fact that, while  $w_\beta(\tau)$  certainly depends on the distribution of widths, *other characteristics of the spectrum become lumped together in the invariance property contemplated in Wigner's conjecture.*

A numerical verification (using the simulation explained above in relation with Wigner's conjecture) of  $w_\beta(\tau)$  of Eq. (7) is shown in Fig. 4 for the three symmetry classes  $\beta = 1, 2, 4$ : in all cases the agreement is seen to be very good.

To summarize, we have found the statistical distribution of capacitances  $p_{\beta,\eta}(\alpha)$ , Eq. (8),  $\alpha$  being defined in Eqs. (4) and (5), for the system shown in Fig. 1, whose essential element is a mesoscopic capacitor. The plate coupled to the back gate is a chaotic cavity; the experimentally relevant situation of one open channel ( $N = 1$ ) is considered and the possibility of direct reflection by the cavity is neglected. The essential ingredient that is needed is the statistical distribution  $w_\beta(\tau)$ , Eq. (7), of time delays  $\tau$  associated with the scattering from the cavity. It is shown that  $w_\beta(\tau)$  can be obtained in a very simple way from a conjecture by Wigner, whose validity, in turn, is verified numerically for the three symmetry classes: orthogonal, unitary, and symplectic. The resulting  $w_\beta(\tau)$  compares very well with the results of numerical simulations, for the three classes. The statistical analysis of the admittance of mesoscopic conductors provides additional information on such systems not contained in the investigation of dc transport properties, and thus points to an interesting avenue of future research.

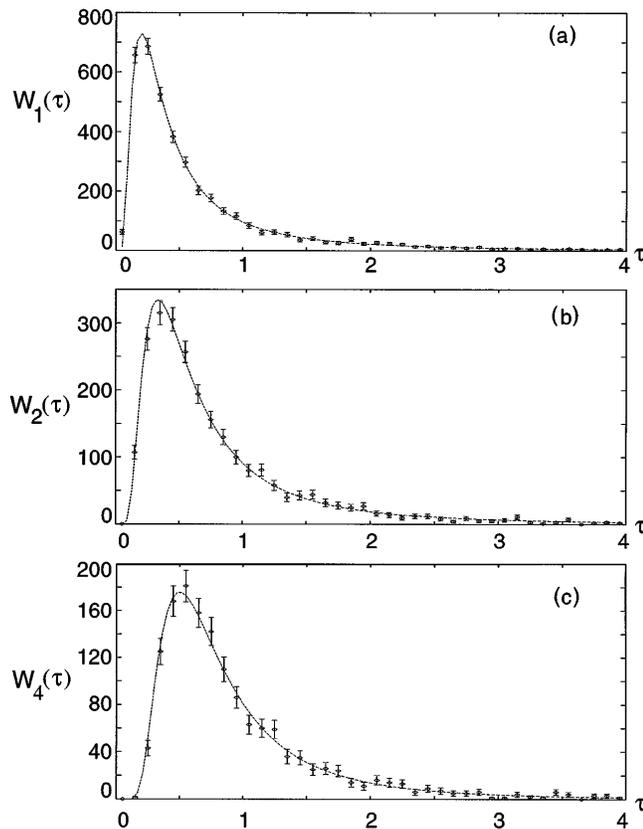


FIG. 4. The distribution  $W_\beta(\tau)$  of time delays for one channel and in the absence of direct processes, for the (a) orthogonal, (b) unitary, and (c) symplectic universality classes. The dotted curves are proportional to the theoretical probability density given by Eq. (7). The points with the finite-sample error bar are the results of the numerical simulation described in the text: 200-dimensional matrices were used in the three cases. The agreement is excellent.

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- [17] The integral of the LHS of (16) is 1 from normalization; the fact that the integral of the RHS is also 1 means  $\langle u \rangle_{\hat{Q}} = 1$ , which is consistent with  $t = 1$ , as required right after Eq. (13). Similarly, from the normalization of  $\hat{Q}(u)$ , Eq. (16) shows that  $\langle 1/u \rangle_{\hat{P}} = 1$ , and hence  $\langle \tau \rangle_w = 1$ .