Chaotic Focusing Billiards in Higher Dimensions

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Numerical computations of Lyapunov exponents for a class of three- and four-dimensional billiards whose boundary consists of flat and spherical components illustrate that such billiards are chaotic due to a defocusing mechanism similar to the one which produces chaos in two-dimensional billiards (e.g., in the stadium). These results demonstrate that recently established rigorous results or higher dimensional defocusing billiards are valid under substantially weaker assumptions. [S0031-9007(96)01233-1]

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In spite of their somewhat unphysical nature, billiards occupy a central position in the physics of dynamical systems. They provide clean mathematical models for the identification of dynamical properties leading to classical chaos; moreover, their quantization yields an ideal testing ground for the semiclassical analysis of quantum systems which have a chaotic classical limit. Their importance in this connection is further enhanced by the possibility of concrete experiments on microwave resonators [1] and on quantum dots [2].

Up to recent times, much attention has been devoted to two-dimensional billiards. In particular, the "stadium" billiard has been extensively used as a paradigm of classical chaos. Chaos arises in the stadium due to a mechanism of defocusing, which was discovered more than 20 years ago [3,4] and was later shown to work also in geodesic flows [5,6]. It is now known [7] that only two mechanisms can generate chaos in Hamiltonian systems; one is dispersion, which works in dispersing billiards as well as in geodesic flows on manifolds of negative curvature, and the other is defocusing, which works in focusing billiards as well as in some geodesic flows on surfaces of non-negative curvature. These two mechanisms enforce local exponential divergence of trajectories, which in turn generates a global chaotic behavior. The dispersing mechanism ensures permanent divergence of nearby trajectories; the mechanism of defocusing is based on strong focusing, which occurs upon a reflection. The beam of rays becomes strongly focused, and, after a short time, passes through a conjugate focusing point. If the free path is long enough, then such a beam becomes divergent, and the time during which nearby rays diverge is, on the average, longer than the time during which they converge. This leads to local exponential instability, and to nonvanishing Lyapunov exponents. If these two mechanisms do coexist in some dynamical system, then the global behavior can be completely chaotic (mixing [3]) but islands of stability can also exist [8]. Rather than enhancing each other, the two mechanisms, when they coexist in the same system, tend to inhibit each other [3,8], and this shows that they are intrinsically different. Whereas the mechanism of dispersion is known to work in dynamical systems of any dimension [9–12], until recently the mechanism of defocusing had only been proven to work in dimension two. An important problem was therefore whether or not defocusing can produce chaos also in a higher dimensional system. In simple words, the question is whether a 3D analog of the stadium billiard exists. In Ref. [13] a focusing "barrel" billiard was studied, which was shown to possess a mixed phase space, with stable and chaotic regions. In Ref. [14] the construction of higher dimensional chaotic focusing billiards was outlined; recently, these ideas were turned into proofs for a class of three-dimensional billiards. The purpose of this paper is to demonstrate the complete chaoticity of the corresponding class of nowhere dispersing billiards in dimensions higher than two, and to show that the geometrical conditions imposed in [15] are indeed too restrictive, as it was surmised in that paper.

We consider billiards in a region Q, whose boundary ∂Q consists of flat and of spherical components. Each spherical component (spherical cap) S of the boundary is characterized by its (internal) angle $\omega(S)$ and by the radius $R(S)$, where $\omega(S)$ is the maximum angle under which two points of the cap are seen from the center of the sphere containing the cap; the angle 2π corresponds to the entire sphere. Suppose that some billiard trajectory has a sequence of consecutive reflections from a spherical cap S, and let these reflections occur at the points $q_1, q_2, \ldots, q_n \in S$. Then all these points together with the center of the sphere containing S belong to the same two-dimensional plane. The dynamics in this plane is analogous to the one in a two-dimensional chaotic focusing billiard. The new feature is the dynamics in the planes that are orthogonal to this one. The dynamics in these "transversal" planes is essentially different: in particular, the focusing in transversal planes is much weaker than in the plane that contains the points of consecutive reflections from a spherical cap. Whereas in dispersing billiards trajectories diverge in all directions after each reflection, in focusing billiards divergence occurs only in a two-dimensional plane; this indicates that chaos in higher dimensional focusing billiards is weaker than in dispersing ones. In order to get nonzero Lyapunov exponents, infinitesimal surface elements normal to billiard trajectories must expand as the trajectories approach spherical caps; moreover, the curvatures of all their two dimensional sections must be sufficiently small. Exact conditions for this have been given in terms of *essential free path* [15]. Such a path is defined as any segment of a billiard trajectory, with reflection points $P_1, \dots P_n$, such that the following conditions are satisfied: (1) The points P_1, P_n belong to spherical caps (not necessarily different). (2) If $n > 2$, all the intermediate points $P_2, \dots, P_{n-1}$ belong to the flat components. (3) If $n = 2$, then P_1, P_2 belong to different spherical caps.

Hence an essential free path contains either reflections from different spherical caps, or reflections from the same spherical cap together with at least one intermediate reflection from some other components of ∂Q. The following theorem was proven in [15].

Theorem: *Suppose that the internal angles of all spherical caps do not exceed $\pi/2$. Then there exists $L = L(Q)$ such that, if the length of every essential free path is larger than L, the billiard in Q has nonvanishing Lyapunov exponents in a subset of full measure of phase space.*

It has been mentioned in [15] that this theorem should be true under milder conditions. However, it does not seem that simple geometrical conditions can ensure nonvanishing Lyapunov exponents in regions with large spherical caps [16].

In this paper we numerically demonstrate the existence of chaos in such regions. To this end we have considered two continuous families of regions, with boundaries consisting of flat components and of spherical caps, which will be described below. For billiards in each family we have numerically computed Lyapunov exponents, by means of the following standard technique. Let z denote a point in six-dimensional phase space, and let $F_t(z)$ be the image of z under the billiard dynamics after the time t^+. Let z_i $(i = 1, 2, \dots)$ be the phase-space point immediately after the ith collision with the boundary, t_i the time of free flight between the ith and the $(i+1)$th collision, and $T_i = \sum_0^{i-1} t_k$ the time at which the ith collision occurs. The tangent map $D_{z_1} F_{T_i^+}$ can be factorized as $\prod_1^i \mathcal{T}_j$, where $\mathcal{T}_j = D_{z_j} F_{t_j^+}$. The linear maps $\mathcal{T}_i$ can be explicitly written in the form of Jacobian matrices as soon as z_i, z_{i+1}, t_i are known. The form of the matrices depends on whether the $(i+1)$th collision takes place on a flat or on a spherical part of ∂Q. In the former case, letting x_j, v_j $(j = 1, 2, 3)$, the position and velocity coordinates of z_i, and X_j, V_j those of z_{i+1}, the matrix elements of $\mathcal{T}_i$ are

$$\frac{\partial X_j}{\partial x_k} = \delta_{jk} - 2n_j n_k, \qquad \frac{\partial X_j}{\partial v_k} = t_i \frac{\partial X_j}{\partial x_k},$$

$$\frac{\partial V_j}{\partial x_k} = 0, \qquad \frac{\partial V_j}{\partial v_k} = \frac{\partial X_j}{\partial x_k} + t_i \frac{\partial V_j}{\partial x_k},$$

where n_j are components of the unit (inward) normal vector to ∂Q at the $(i+1)$th collision point. If the collision occurs on a spherical component, of radius r, the $3D$ equation must be changed to

$$\frac{\partial V_j}{\partial x_k} = \frac{2}{r}(v_n \delta_{jk} + n_j v_k - n_k v_j - n_j n_k),$$

where v_n is the normal component of velocity; the fourth equation changes accordingly. By numerically computing one billiard orbit, one can find the tangent map at all collision times by numerically computing a product of Jacobian matrices of rank 6. On applying the tangent map thus determined to a triple of orthonormal tangent vectors, and using the well-known method of intermediate orthonormalizations [17], one is able to compute all three non-negative Lyapunov exponents in real physical time. One of these, which correspond to infinitesimal displacements along the trajectory, must vanish; its numerical value of $\sim 10^{-3}$ after 10 000 collisions provides a lower bound for the numerical error involved in our computations.

The first family of billiards is defined in regions $Q_r = K \cap S_r$, where K is a fixed cube with edges of length 2 and center O, and S_r is a sphere of radius r and with the same center O. The radius r of the sphere is the parameter of the family. For $0 < r < 1, Q_r = S_r$, while, for $r > \sqrt{3}, Q_r = K$. These extreme cases correspond to integrable billiards. In between these extreme cases, the region Q_r looks like a "fair die" and is depicted in Fig. 1, for the intermediate ranges of parameters $1 < r < \sqrt{2}$ and $\sqrt{2} < r < \sqrt{3}$. In the case of Fig. 1(a) the billiard has no "caps," because the spherical part of the boundary consists of one connected component.

In both cases 1(a) and 1(b) typical 3D trajectories, which get reflected from the flat as well as from the spherical parts of the boundary, have two positive Lyapunov exponents λ_1, λ_2 $(\lambda_1 > \lambda_2)$, the dependence of which on the radius r is shown in Fig. 2. For every different value of the radius, a single trajectory was used in the computation, starting from the same point in Q_r, but with a different, randomly chosen velocity of modulus 1. The exponents shown in Fig. 2 correspond to 10 000 collisions. At $r = 1.2$, exponents λ_1, λ_2 computed after 40 collisions lie within 0.01, 0.03, respectively, of their final values. At $r = 1.5$, stabilization within 0.015 of the final value is achieved after 250 collisions, for both exponents.

On computing the Lyapunov exponents on a sample of 10^2 randomly chosen trajectories, at fixed r, a rms statistical dispersion was found, ranging from 4×10^{-3} to 8×10^{-3}; the latter value is attained close to the integrable limits, where long integrable segments appear in a typical orbit.

Besides truly 3D orbits, also "planar" trajectories exist, which are bound to fixed planes and are therefore identical to trajectories of some 2D billiard. Such orbits have a different type of stability than discussed above. For

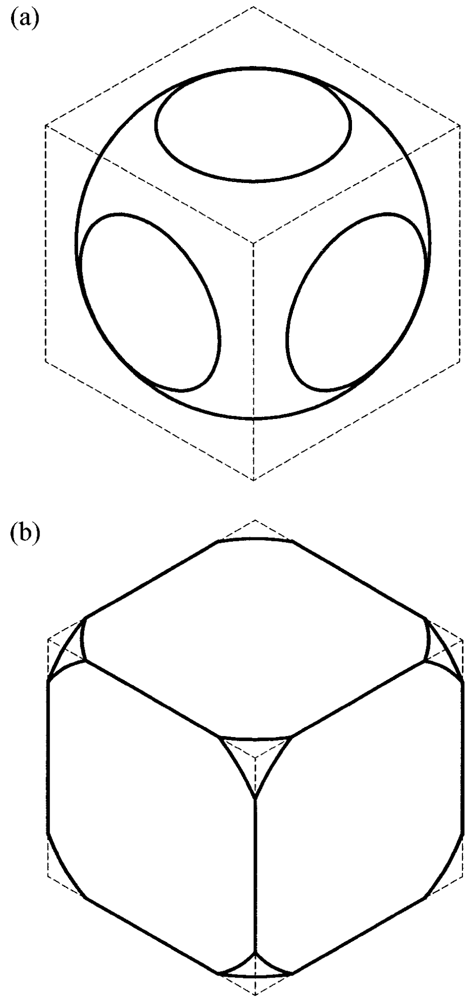

FIG. 1. The family of 3D "dice" billiards: (a) $1 < r < \sqrt{2}$; (b) $\sqrt{2} < r < \sqrt{3}$.

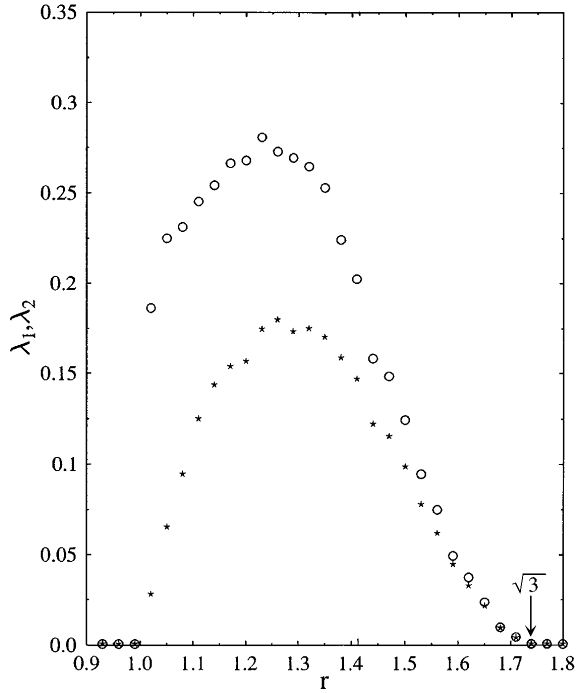

FIG. 2. Lyapunov exponents λ_1, λ_2 for 3D orbits in 3D dice billiards versus the radius r of the sphere.

$1 < r < \sqrt{3/2}$ planes exist through the center O, which do not intersect any of the flat components. Trajectories which start in such a plane are bound to it forever, and are, in fact, trajectories in a circular billiard. They are linearly stable (all Lyapunov exponents vanish). Such orbits occupy an invariant region of phase space, which has a positive (normalized) measure which decreases from 1 at $r = 1$ to 0 at $r = \sqrt{3/2}$. The existence of this stable region makes the billiard a nonergodic one in the corresponding parameter range.

Other planar trajectories are bound to planes which contain one of the orthogonal symmetry axes of the cube. These planes can be of two types: (i) those which have a normal intersection with two of the flat components, and no intersection at all with the other flat components, and (ii) those which actually contain two orthogonal symmetry axes. Planar trajectories of type (i) only exist in 1(a) [in case 1(b) they are singular, because they hit the edges of the cube], and are trajectories of a 2D billiard of the stadium type. They make a set of measure 0, and have only one positive exponent. Planar trajectories of type (ii) also have 0 measure; for $1 < r < \sqrt{2}$ they have two nonvanishing exponents that can be significantly larger than the exponents of generic orbits; e.g., at $r = 1.1$ we found $\lambda_1 \approx 0.3$, to be compared with the value $\lambda_1 \approx 0.2$ for generic orbits. At $r > \sqrt{2}$ type (ii) trajectories become trajectories in a square 2D billiard, and are linearly stable.

Our results show that invariant hyperbolic subsets appear in high dimensional focusing billiards under much milder conditions than were imposed in the above theorem. The case of "fair dice" demonstrates that it might be sufficient that the essential free path should be not less than a corresponding chord of the sphere that contains the spherical cap in the boundary. It is easy to see that this condition is always a necessary one; indeed, a period two trajectory that is perpendicular to two spherical caps is linearly stable if its length is less than the diameter of these spheres.

The family of "dice" billiards has an analog in any number of dimensions. Results for the Lyapunov exponents $\lambda_1, \lambda_2, \lambda_3$ of a 4D die, versus the radius r of the sphere, show a behavior qualitatively similar to the 3D case (Fig. 3). The "cubic" limit is now attained at $r = 2$. The points closest to $r = 1$ have apparently fallen into the stable region.

The second series of our numerical experiments has the goal to check how big a spherical cap in a chaotic focusing billiard can be. Billiards are now defined in regions $Q_z = K \cup S_z$, where K is the same cube as before, and S_z is a sphere of constant radius 0.5, with center on the vertical line through the center O, at a distance $z < 1.5$ from O. If $0 < z < 0.5$ one gets the billiard in the cube K. If $0.5 < z < 1.5$ the billiard region is a "dome," that

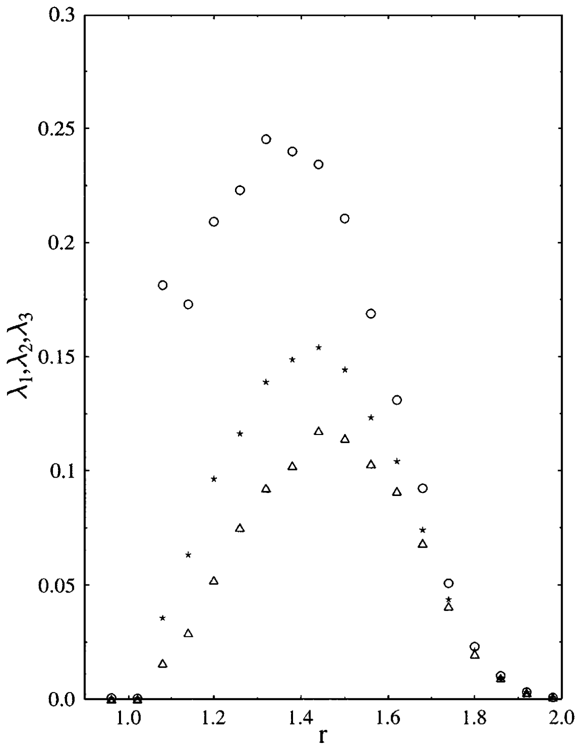

FIG. 3. Lyapunov exponents $\lambda_1, \lambda_2, \lambda_3$ for 4D orbits in 4D dice billiards versus the radius r of the sphere.

is, a cube with a spherical cap on the top. In the case $1/2 < z < 1 - 1/2\sqrt{2}$ the internal angle of the spherical cap is less than $\pi/2$; this case was considered in [15].

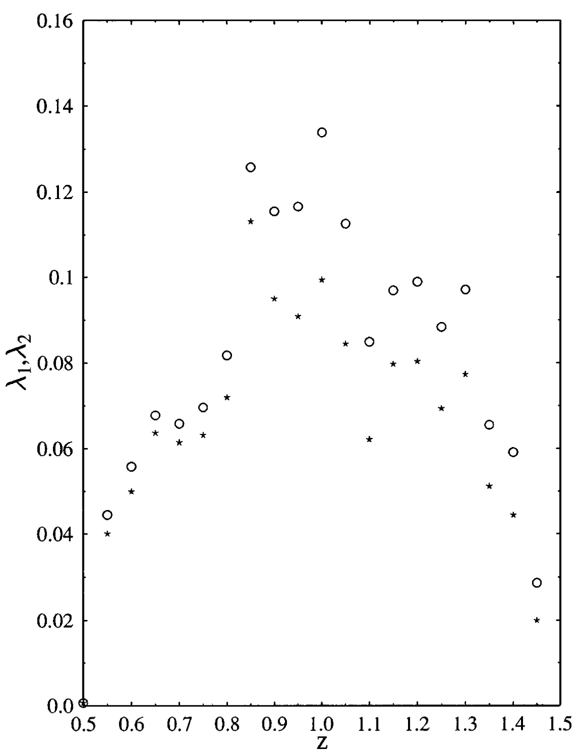

FIG. 4. Lyapunov exponents λ_1, λ_2 for the "dome" billiards versus the coordinate z of the center of sphere.

However, in the whole region $1/2 < z < 3/2$ we have found two positive Lyapunov exponents, $\lambda_1 > \lambda_2 > 0$ as shown in Fig. 4. The rms dispersion over an ensemble of 100 different trajectories was in this case $\sim 2 \times 10^{-2}$. Stabilization of numerical Lyapunov exponents for single trajectories within 0.03 of their final values typically requires $\sim$1000 collisions at $z = 0.75$.

"Planar" trajectories, which lie in fixed planes through the center of the cube, parallel to two faces, also display two positive exponents, typically larger than the exponents of generic orbits; at $z = 0.85$ the maximal exponent of such an orbit is $\lambda_1 \approx 0.25$. It is worth mentioning that the shortest unstable periodic orbit, bouncing along the vertical axis between the cap and the opposite face of the cube, has still larger, practically coincident positive exponents ($\approx$0.6 at $z = 0.8$).

Therefore our analysis revealed that the mechanism of defocusing may generate a chaotic behavior in billiards under rather mild conditions. More precisely, spherical regions of the boundary can be arbitrarily large (that is, ∂Q can be arbitrarily close to a sphere), and essential free paths can be bounded from below just by the length of the corresponding spherical cord.

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