Equivalence of Synchronization and Control of Chaotic Systems

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It is shown that "perfect" control of a system along a desired trajectory is the control analog of the different techniques of synchronization of chaos. Analysis of this control problem leads to a general framework for synchronization of chaotic systems. Numerical examples are presented to illustrate the connection between synchronization and perfect control. Modifications of the control scheme are carried out to make the control technique useful in practical situations. [S0031-9007(96)01174-X]

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Since 1990 synchronization of chaos has been a topic of great attention and puzzlement. Though commonly observed, synchronization between two identical chaotic systems has been thought to be an unlikely goal because chaos is characterized by a sensitive dependence on initial conditions. Beginning with the work of Pecora and Carroll (PC) [1,2] several investigations have been carried out on different aspects of synchronization and on developing its applications [3,4]. Kocarev and Parlitz [3] have investigated a general approach for constructing chaotic synchronized dynamical systems which they refer to as the method of active-passive decomposition (APD). Recently Rulkov *et al.* [5] and Kocarev and Parlitz [4] have investigated another generalized approach to synchronization. This is referred to as generalized synchronization (GS). Here two systems are said to synchronize if a functional relationship exists between the states of both systems. It is known that synchronization based on APD and GS are generalizations of PC synchronization (PCS).

In this Letter, we present a general framework for the analysis of different types of synchronization of chaos. We show that synchronization is another interpretation of the use of feedback to achieve "perfect" control of a process along a desired trajectory. By perfect control we mean that the controlled output tracks the desired output exactly for all time $t \geq 0$. We show that the three types of synchronization mentioned above are different cases of perfect control. The connection between synchronization and perfect control is illustrated on a modified form of the Rossler system. In the second part of this Letter we reconsider the perfect control problem and derive a practical control scheme.

Figure 1 shows a schematic of the problem under consideration. System 1 is a model representing the desired behavior of the process (reference system) and system 2 represents the process. It is desired to match (synchronize) the evolution of the controlled process output along its desired trajectory for time $t \geq 0$ using feedback. Our objective is different from the conventional control goal where it is desired that synchronization occurs as $t \to \infty$. In general perfect control is not a desirable objective because we require arbitrarily large parametric perturbations to achieve the goal and we need to fix the initial states of some of the system states.

Next we outline the formulation and solution of the control problem.

Recent developments in nonlinear control theory [6–9] enable us to formulate and solve the control problem in a systematic way. We consider the desired output and the process to be represented by the following n-dimensional single input–single output (SISO) models

$$\dot{\mathbf{x}}_{\mathbf{m}} = \mathbf{f}(\mathbf{x}_{\mathbf{m}}) + \mathbf{g}(\mathbf{x}_{\mathbf{m}})u_0\,, \tag{1}$$

$$y_d = h(\mathbf{x}_{\mathbf{m}})\,, \tag{2}$$

$$\dot{\mathbf{x}}_{\mathbf{p}} = \mathbf{f}(\mathbf{x}_{\mathbf{p}}) + \mathbf{g}(\mathbf{x}_{\mathbf{p}})u\,, \tag{3}$$

$$y = h(\mathbf{x}_{\mathbf{p}})\,. \tag{4}$$

$\mathbf{x}_{\mathbf{m}}, \mathbf{x}_{\mathbf{p}} \in \mathbf{R}^n$ are the reference and process states, u_0 and u represent the nominal and the manipulated input, respectively. We further assume the inputs to occur in only one equation. y is the scalar measured output and y_d denotes the scalar desired output to be tracked. Our objective is to apply suitable parametric perturbations such that the process output tracks the desired output exactly, i.e.,

$$y_d - y = 0 \quad \text{for } t \geq 0\,.$$

The relation between the manipulated input u and the process output y can be expressed as

$$u = \frac{y^r - L_{\mathbf{f}}^r h(\mathbf{x}_{\mathbf{p}})}{L_{\mathbf{g}} L_{\mathbf{f}}^{r-1} h(\mathbf{x}_{\mathbf{p}})}\,, \tag{5}$$

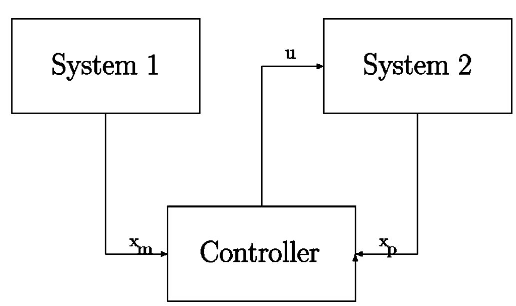

FIG. 1. Schematic illustration of the feedback control strategy.

where y^r is the rth order derivative of the process output, r is a measure of how directly the input affects the output. It is known as the relative order of the system and is defined as the smallest integer such that

$$L_{\mathbf{g}} L_{\mathbf{f}}^{r-1} h(\mathbf{x_p}) \neq 0 .$$

$L_{\mathbf{f}} h(\mathbf{x_p})$ is the Lie derivative of a function $h(\mathbf{x_p})$ with respect to $f(\mathbf{x_p})$ and is defined as

$$L_{\mathbf{f}} h(\mathbf{x_p}) = \sum_{k=1}^{n} \mathbf{f}_k(\mathbf{x_p}) \frac{\partial h(\mathbf{x_p})}{\partial x_k} .$$

Higher order Lie derivatives with the same vector argument are defined recursively as

$$L_{\mathbf{f}}^{m} = L_{\mathbf{f}} L_{\mathbf{f}}^{m-1} .$$

For the problem under consideration, the measured and the desired outputs are specified by the designer. The control law which achieves the objective of perfect control is given by replacing y^r in (5) by y_d^r. It is obvious that requesting perfect control constrains the process output to track its desired trajectory exactly. The question of determining whether the system states also approach their desired trajectories will be addressed a little later.

We digress to mention how the three types of synchronization result from consideration of the control problem. Later we present examples to show that synchronization is another interpretation of perfect control when $r = 1$. The case considered by PC [1] results from the case when both the desired and measured outputs are states ($y_d = x_{mi}$ and $y = x_{pi}$, $1 \leq i \leq n$). GS [4,5] results from considering the desired output as a function of states $y_d = h(\mathbf{x_m})$ and $y = h(\mathbf{x_p}) = x_{pi}$, $1 \leq i \leq n$. For the case of synchronization using APD we rewrite the system (after neglecting the subscripts) as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u + \mathbf{e}(\mathbf{x}) - \mathbf{e}(\mathbf{x}) ,$$

where $\mathbf{e}(\mathbf{x})$ is some function of the states $\mathbf{x}$. We can define a new variable x_{n+1} as the complete (or part of) function $\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u + \mathbf{e}(\mathbf{x})$. Further, we can represent the evolution of this new variable by an arbitrary equation with a new input u_{new}. The model and process are now given by equations of the form

$$\dot{\mathbf{x}} = \overline{\mathbf{f}}(\mathbf{x}) + \overline{\mathbf{g}}(\mathbf{x}) x_{n+1} - \mathbf{e}(\mathbf{x}) ,$$

$$\dot{x}_{n+1} = d(\mathbf{x}) + u_{\text{new}} .$$

Now consider the case where the desired and process output are given by $y_d = x_{m,n+1}$ and $y = x_{p,n+1}$ and the manipulated input is u_{new}. Perfect control for this case is the control analog of synchronization using APD. Examples of this case can be found in Table I of Parlitz *et al.* [10].

Returning to the control problem, information about internal stability of the process (i.e., whether the process states also approach their desired trajectories) can be obtained from a stability analysis of the zero dynamics of the process. Next we outline the method for obtaining the zero dynamics of a system. Systems with asymptotically stable zero dynamics are said to be minimum phase.

Further analysis is simplified by working in transformed coordinates. Noting that $L_{\mathbf{f}}^{k} h(\mathbf{x_p}), k = 0, \ldots, r-1$ are linearly independent functions of $\mathbf{x_p}$, we can choose these to be the first r elements in defining a transformed cooordinate system $\mathbf{z}$, i.e.,

$$z_k = L_{\mathbf{f}}^{k-1} h(\mathbf{x_p}), \qquad k = 1, 2, \ldots, r .$$

Moreover, it is possible to choose $(n - r)$ additional coordinates $z_k = \phi_k(\mathbf{x_p})$, $k = (r+1), \ldots, n$, such that their time derivatives are independent of u [8,9]. In the $\mathbf{z}$ coordinates, the SISO nonlinear system can be represented in its normal form

$$\begin{aligned}
\dot{z}_1 &= z_2 & \dot{z}_{r+1} &= q_{r+1}(\mathbf{z}) \\
&\vdots & &\vdots \\
\dot{z}_r &= b(\mathbf{z}) + a(\mathbf{z})u & \dot{z}_n &= q_n(\mathbf{z}) \\
& & y = z_1 ,
\end{aligned}$$

where

$$a(\mathbf{z}) = L_{\mathbf{g}} L_{\mathbf{f}}^{r-1} h[\phi^{-1}(\mathbf{z})], \qquad b(\mathbf{z}) = L_{\mathbf{f}}^{r} h[\phi^{-1}(\mathbf{z})] ,$$

and

$$q_k(\mathbf{z}) = L_{\mathbf{f}} \phi_{k+r}[\phi^{-1}(\mathbf{z})], \qquad k = 1, \ldots, n - r .$$

A minimal-order realization of the inverse (MORI) of the system is obtained by replacing u in the normal form above by (5), after a suitable change of coordinates. The advantage of using the normal form representation of nonlinear SISO system is that its inverse is effectively of dimension $(n - r)$. The zero dynamics are the dynamics of a MORI, and they represent the system dynamics when the system output is constrained to be the desired output.

We now illustrate the connection between perfect control and synchronization with the help of an example. We consider a modified form of the Rossler system. We represent the reference system (system 1) and the process (system 2) as

$$\dot{X}_1 = -Y_1 - Z_1 , \tag{6}$$

$$\dot{Y}_1 = X_1 + aY_1 , \tag{7}$$

$$\dot{Z}_1 = b + Z_1(X_1 - c) , \tag{8}$$

$$\dot{X}_2 = -Y_2 - Z_2 + p_1 , \tag{9}$$

$$\dot{Y}_2 = X_2 + aY_2 + p_2 , \tag{10}$$

$$\dot{Z}_2 = b + Z_2(X_2 - c) + p_3 , \tag{11}$$

with $a = b = 0.20$ and $c = 9$. In what follows, we also assume that all the auxiliary inputs (p_1, p_2, p_3) excluding the manipulated input are set to zero.

We consider the case where $y = X_2$, $u = p_1$ (with $p_2 = p_3 = 0$), and $y_d = X_1$. Note that the manipulated input occurs only in the governing differential equation of the controlled output and hence $r = 1$. For perfect control, we require that $y = y_d$. This means that at $t = 0$, we require $X_2 = X_1$. The control law for tracking the desired trajectory y_d is

$$u = p_1 = \dot{y}_d - (-Y_2 - Z_2). \tag{12}$$

Substituting this back in the equations of the process (8)–(11), we get

$$\dot{X}_2 = \dot{y}_d, \tag{13}$$

$$\dot{Y}_2 = y_d + aY_2, \tag{14}$$

$$\dot{Z}_2 = b + Z_2(y_d - c). \tag{15}$$

It is evident that by applying the above control law, the desired output can be tracked exactly. We now address the issue of internal stability of the process. Since the desired trajectory is chaotic, the necessary condition for asymptotic stability of the zero dynamics [Eqs. (14) and (15)] is that its Lyapunov exponents [1] must be negative. These were calculated to be $+0.20$ and -8.87. This indicates that the zero dynamics is unstable (the system is nonminimum phase) and hence the process is not internally stable, i.e., the variables Y_2 and Z_2 do not approach their respective desired trajectories. An example of GS for this case results from taking $y_d = X_1 + Y_1 + Z_1$. An example of synchronization using APD would be to define a new variable $x_{n+1} = s = 1.2Y_1$ and then rewrite the equation for Y as $\dot{Y}_1 = X_1 + aY_1$ and $\dot{Y}_2 = X_2 - Y_2 + s$. The control objective for this case is $y = s$. More examples can be found in [10].

The set of equations (6)–(8), (14), and (15) is identical to the case of the X drive in the PC approach. This is because the control parameter p_1 occurs only in the equation of the controlled output X_2. Also the subsystem defined by them is identical to the notion of MORI (and the zero dynamics) in the control literature. Similarly PCS using Y drive and Z drive can be shown to be identical to requesting perfect control of Y_2 using p_2 as the manipulated input and perfect control of Z_2 using p_3 as the manipulated input, respectively. This example also brings out the limitations of the PCS and GS schemes. In these two approaches the natural zero dynamics of the system are not altered. This is not the case with synchronization using APD. In this case the additional variable x_{n+1} can be selected such that the zero dynamics are always asymptotically stable [10].

To summarize, we have shown synchronization of chaos to be a specific case of perfect control of a process of relative order one. In the physics literature perfect control is interpreted as injecting the desired signal into a subsystem of the original system. The condition which must be satisfied in order for synchronization to occur is that the system must be minimum phase. This is a generalization of the stability conditions currently employed, i.e., computing the conditional Lyapunov exponents of the response subsystem.

We return to the problem of perfect control of the process when the manipulated input does not occur in the governing equation of the controlled output. We illustrate the method by considering X_2 to be the controlled output ($y = X_2$ and $y_d = X_1$) and p_2 to be the manipulated input. For this case, we have $r = 2$. Proceeding as before, the control law can be obtained as

$$u = p_2 = -\ddot{y}_d - \dot{Z}_2 - y_d - aY_2. \tag{16}$$

Substituting the control law back in the equations describing the process (8)–(11), we get

$$\dot{X}_2 = \dot{y}_d = \dot{X}_1, \tag{17}$$

$$\dot{Y}_2 = -\ddot{y}_d - \dot{Z}_2 = \dot{Y}_1 + \dot{Z}_1 - \dot{Z}_2, \tag{18}$$

$$\dot{Z}_2 = b + Z_2(y_d - c). \tag{19}$$

Requiring perfect control on initiating the process setting $X_2 = X_1$ ($y = y_d$) implies $\dot{X}_2 = \dot{X}_1$. On rearranging, this results in $Y_2 = -\dot{y}_d - Z_2$. This implies that manipulating p_2 in order to make X_2 track X_1 constrains Y_2 to evolve such that the above relation is satisfied. The internal stability of the process is now dependent on the asymptotic stability of the remaining equation (19), which represents the zero dynamics for this case.

Lyapunov exponents of the zero dynamics for each combination of the input-output (output = state) are given in Table I(a). These are analogous to the conditional Lyapunov exponents of [1,2]. Nondiagonal cases for which we have only one Lyapunov exponent can be considered to be similar to synchronization with a two variable drive in the approach of PC. The absence of zero dynamics for the $Y_2 - p_3$ and $Z_2 - p_2$ configurations means that the Lyapunov exponents are equal to zero for these cases [7].

Similar results were obtained with a similarly modified form of the Lorenz system. The reason we chose to work with modified forms and not with the original equations becomes evident when we consider the case of tracking Y_2 using p_1 ($y_d = Y_1$). The control law is

$$u = p_1 = \frac{\ddot{y}_d + \dot{y}_d + X_2\dot{Z}_2}{r - Z_2} - \sigma(y_d - X_2). \tag{20}$$

The control law is not applicable when the denominator is zero. Using the modified Lorenz equations enables us to obtain proper forms for the control law for four combinations of the input-output (output = state). Table I(b) gives the Lyapunov exponents for the zero dynamics for each case. The diagonal cases correspond to

TABLE I. Lyapunov exponents of the zero dynamics for the process system for different control configurations of the (a) Rossler system and (b) Lorenz system for $\sigma = 10$, $r = 60$, and $b = 8/3$.

	p_1	p_2	p_3		p_1	p_2	p_3
		(a)				(b)	
X_2	0.20		0.20	X_2	-1.78		···
	-8.87	-8.87			-1.88	-2.667	···
Y_2		-0.024		Y_2	···	-10.0	···
	-8.87	-8.8			···	-2.667	···
Z_2	0.20		0.10	Z_2	···	···	0.014
			0.10		···	···	-11.02

synchronization using X, Y, and Z drives, respectively, in the approach of PC. The only nondiagonal case for which a proper control scheme can be derived ($X_2 - p_2$) is identical to synchronization using two variable (X-Y) driving signals in the approach of PC.

The feedback scheme discussed above suffers from two major drawbacks, the necessity to fix the initial values for some of the process variables and the inability to impose bounds on the manipulated input. We can impose bounds on the input provided we forego our desire for perfect control. The necessity of fixing the initial values of the process variables can be overcome by requesting the controlled output to approach the desired trajectory in a predetermined fashion. For example, by appropriately selecting two new constants α and β such that

$$y_d - y = \alpha e^{-\beta(t-t_0)}. \tag{21}$$

The values of these constants can be evaluated from a knowledge of the initial states of the process variables. Control can be initiated when β takes on positive values. This scheme was modified to achieve (i) perfect control when the manipulated input is within the allowed bounds and (ii) leaving the system to freely evolve when the manipulated input exceeds its maximum permitted bounds. The modified scheme involved resetting the values of α, β, and t_0 when the manipulated input crossed its bounds. The results on the modified Rossler system for different combinations of the input-output are shown in Figs. 2 and 3. The effect of measurement noise on the tracking of the desired trajectory is also represented in the figures. The strong dependence of noise on the relative order is readily seen.

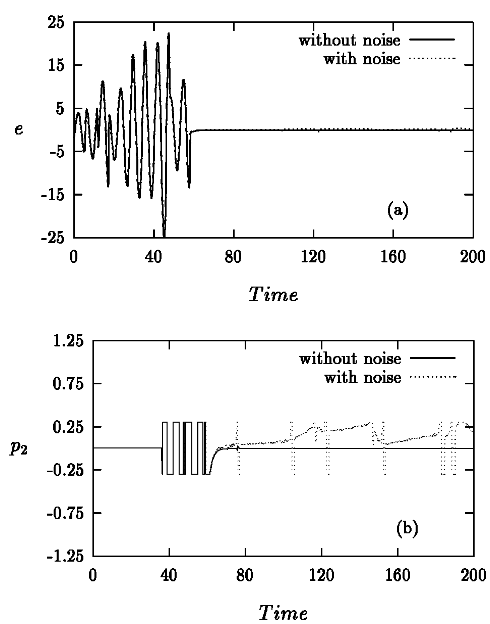

FIG. 2. (a) Variation of the error $e = y_d - y$ with time when $y = X_2$, and (b) variation of the manipulated input p_2 with time for the Rossler system. p_2 is allowed to vary between -0.30 and 0.30.

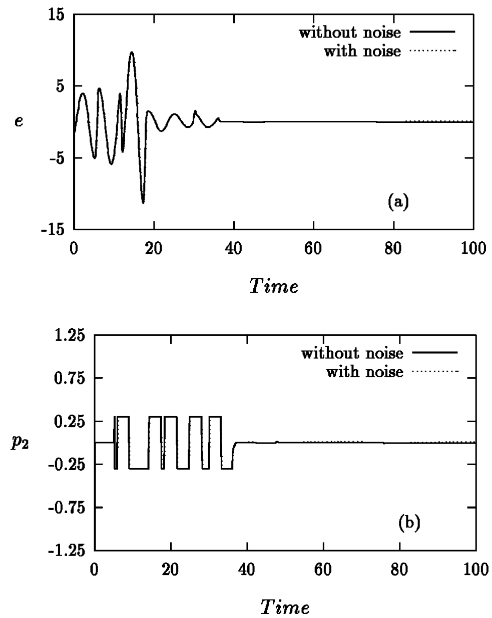

FIG. 3. (a) Variation of the error $e = X_1 - X_2$ with time when $y = Y_2$, and (b) variation of the manipulated input p_2 with time for the Rossler system. p_2 is allowed to vary between -0.30 and 0.30 and $\beta = 5$ in (21).

In conclusion, we have presented a nonlinear control technique to achieve perfect control of a system along a desired trajectory. An important result of this work is that this method is the control analog of the synchronization phenomenon discussed in the literature. In this connection we have presented results from the control literature for the analysis of synchronization in a unified framework.

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