Simulations of Two-Dimensional Kadomtsev-Petviashvili Soliton Dynamics in Three-Dimensional Space

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This Letter reports on three-dimensional simulations that follow exact, z symmetric soliton solutions to an important equation of plasma physics and fluid dynamics. These solitons are seen to break up when perturbed slightly along z. Some fairly robust, three-dimensional entities are subsequently produced. However, they do not seem to resemble known, azimuthally symmetric solutions. An explanation of this interesting fact is offered. [S0031-9007(96)01348-8]

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Solitons are by now well-researched entities that appear both in nature and in present day mathematical considerations of nonlinear, partial differential equations. The 3D Kadomtsev-Petviashvili equation considered here [1] is

$$(n_t + 6nn_x + n_{xxx})_x - 3(n_{yy} + n_{zz}) = 0. \qquad (1)$$

This equation is known as KPI. It describes the dynamics of nonlinear waves and solitons in a variety of media (plasmas, fluids) [2–4] about which more follows. Equation (1) is usually much simpler than the full set of equations it models. In deriving it, one assumes weak dispersion and that the soliton or nonlinear wave in question propagates along x. Changes in y and z are slower than in the direction of motion. Equation (1) is not integrable by inverse scattering unless $\partial_z = 0$. For a more extensive discussion of the derivation of (1) and some solutions, see Ref. [5].

The KPI equation appears in plasma physics when describing small amplitude, fast magnetosonic waves (FMS) propagating in a low β $(= 8\pi p/B^2)$ magnetized plasma. If these waves propagate at an angle with respect to the magnetic field, collapse may occur. Collapse mechanisms can, in turn, lead to a transfer of energy to the plasma ions. A better understanding of these mechanisms is lacking. More generally, the fate of nonlinear structures in plasmas is currently being investigated in several plasma physics laboratories. Certain restrictions, such as small amplitude, long wavelength, propagation velocity within a small interval around that for linear FMS waves, wave frequency small as compared with the ion cyclotron frequency, etc., are assumed when deriving KPI in this context. Nevertheless, it is often subsequently found that, by serendipity, this model equation's usefulness extends far beyond the expected region in parameter space, see Chap. 8 of Ref. [5].

A second area where KPI appears is the condensate model of superfluid helium. The two- and three-dimensional solitons of KPI then model the vortex lines and rings that have been observed there. In this context, KPI is derived as a limit of the nonlinear Schrödinger equation, which itself is a somewhat controversial model for helium II. At present, it would seem that KPI is more firmly grounded in physical reality in the case of FMS waves as compared to the Bose condensate context.

There are also further physical contexts in which KPI appears, e.g., a very thin sheet of water, $h < 0.47$ cm, if surface tension is taken into account [5]. However, physical context for physical context, KPI is not so far as ubiquitous in theoretical physics as its mathematically less interesting companion, KPII [a plus in front of the last term in (1); it has no two-dimensional soliton solutions].

It is of interest to physicists when a soliton collapses in three dimensions and produces a known nonlinear structure. When, however, the resulting structure was not hitherto known, interest is even greater. This is the case described in the present Letter.

Exact, N soliton solutions to both one- and two-dimensional space versions of (1) are well known. The one-dimensional soliton $n(x - vt)$ is unstable in two dimensions [5], and its breakup into an array of two-dimensional, exact soliton solutions has been demonstrated both analytically and numerically [6–8]. The two-dimensional soliton $n(x - vt, y)$ is stable in two dimensions [9].

We now come to the more physical question of what happens in three dimensions. As the 1D soliton, having plane symmetry, is known to disintegrate into 2D entities in two dimensions, it should certainly break up in three dimensions. The 2D soliton $n(x - vt, y)$ is known from theory to be unstable in three dimensions [9,10]. However, perturbing it along the z axis in a simulation, actually seeing it break up and following the debris, demands somewhat cumbersome numerics. As far as we know, this has not yet been done. Breakup of 3D, azimuthally symmetric solitons *has* been investigated numerically [10–12]. It might seem odd that simulations for breakup of the 3D soliton have been performed, whereas for the 2D one have not. The explanation is in the symmetry. The 3D soliton breakup can be studied in $(x, \rho)[\rho = (y^2 + z^2)^{1/2}]$ space, neglecting θ. On the other hand, disintegration of the 2D soliton in three-dimensional space can only be followed in a fully

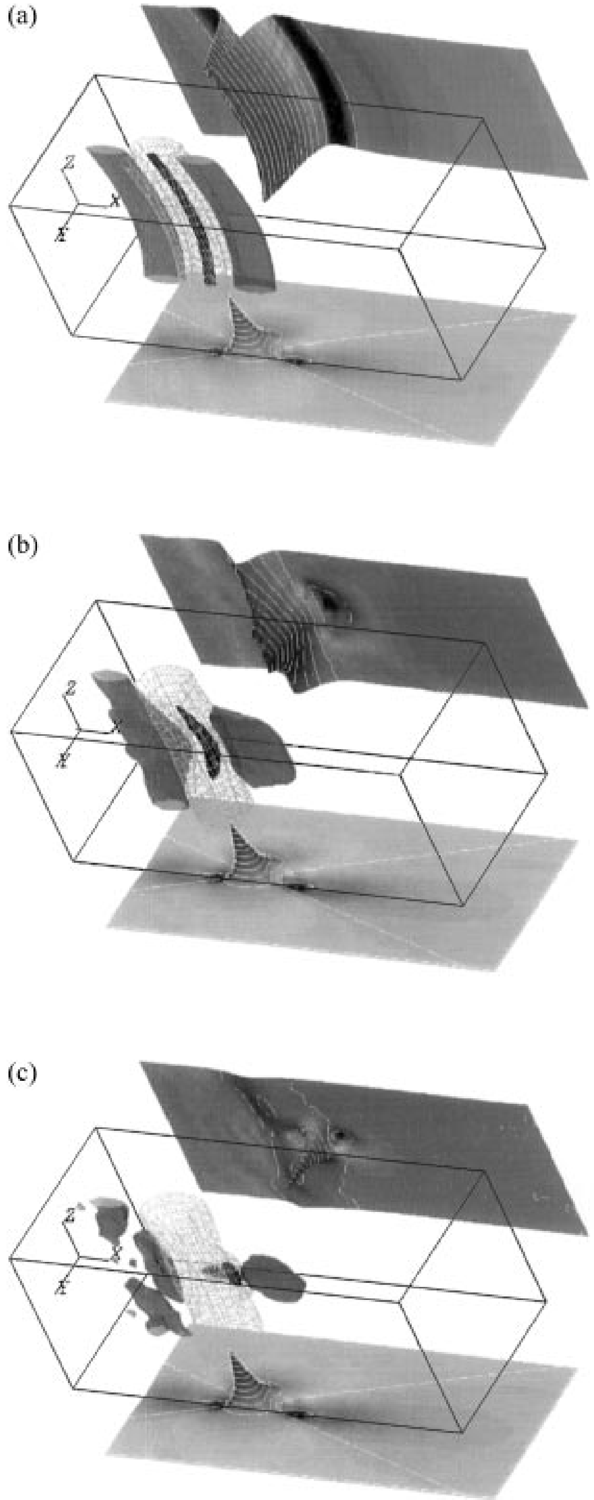

FIG. 1. Consecutive stages of the evolution of a 2D soliton initially perturbed along z. The equation of the soliton at $t = 0$ is given by (2) with x replaced by $x + \delta\cos(k_z z)$. Here $\nu = \frac{17}{6}, \delta = 0.12, k_z = \frac{1}{3}$. Three surfaces of constant n $(= -0.8, 1, 8.84)$ are seen. The three-dimensional soliton slowly breaks up producing new structures and debris. Cross sections along x, y (bottom) and x, z (top) are shown. Initially the bottom ones correspond to constant n lines as given by (2).

three-dimensional simulation. This task is the subject of our Letter.

Although we know from theory that our soliton will disintegrate, there are at least two ancillary questions to be addressed: (1) Will any products of the breakup be robust? (2) If so, will they resemble the 3D soliton solutions, known approximately from the theory [13]?

In fact, a similar 3D simulation was performed for a different model soliton equation [Zakharov-Kuznietsov, which differs from (1) only in the y and z terms [14]]. There the answers to both questions were affirmative. However, in that case, the 3D solitons were known to be stable, in contradistinction to the present study of (1). Thus, until we performed the simulation, we had to consider both questions 1 and 2 as open.

The two-dimensional soliton solution [15] is

$$n(x,y,t) = \frac{4\nu[1 - \nu(x - 3\nu t)^2 + \nu^2 y^2]}{[1 + \nu(x - 3\nu t)^2 + \nu^2 y^2]^2}, \quad \nu > 0. \tag{2}$$

Note that n, which is the *excess* over the mean density in (1), can be negative. Valleys appear around the x axis. As already mentioned, theory tells us that this soliton is unstable in x, y, z, t.

The three-dimensional soliton resembles (2) rotated around the x axis. However, known solutions are approximate [not exactly (2) with $y \to \rho$] [13]. This is not surprising in view of the nonintegrability of (1) in 3D.

Our calculations were performed on an HP Apollo 9000 model 720 workstation. The numerical algorithm used for calculating the time evolution was the leapfrog algorithm, applied when (1) was integrated over x. Periodic boundary conditions were assumed. More about the algorithm and its stability can be found in the second of Refs. [8]. These 3D calculations took several hundred computer hours.

The soliton (2) was wiggled along z such that, at $t = 0$, x was replaced by $x + \delta\cos(k_z z)$. Since we do not know the value of k_z corresponding to maximum growth rate of the instability, the "wave number" was chosen by trial and error. The Kadomtsev-Petviashvili equation is peculiar in that even the initial conditions must fulfill an infinite set of constraints. The most obvious one follows from an x integration:

$$\nabla_\perp^2 \int_{-\infty}^{\infty} n dx = 0.$$

Our initial condition satisfies all these constraints. Again, this aspect is discussed extensively in [8] (for 2D; see also [16] for theory). Figures 1–3 show three-dimensional visualizations of various constant n surfaces as the soliton propagates. Two cross sections augment each frame. The

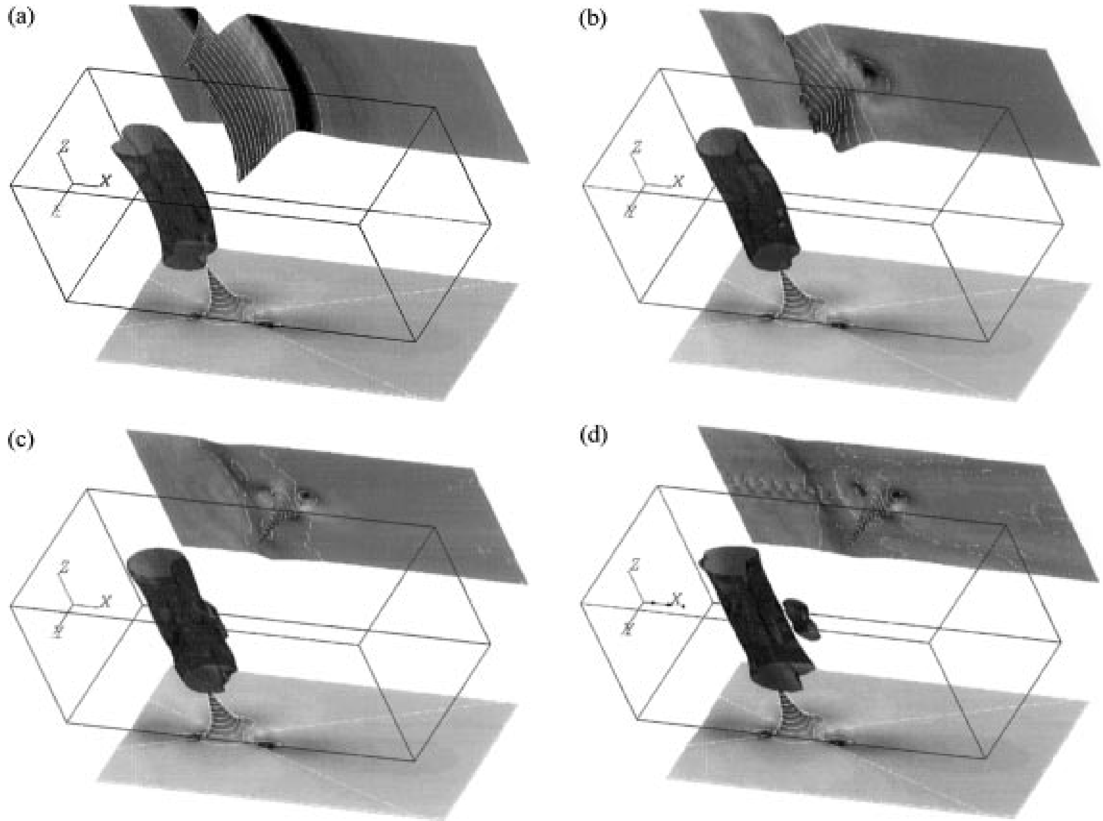

FIG. 2. As in Fig. 1, but only the middle, $n = 1$, surface displayed.

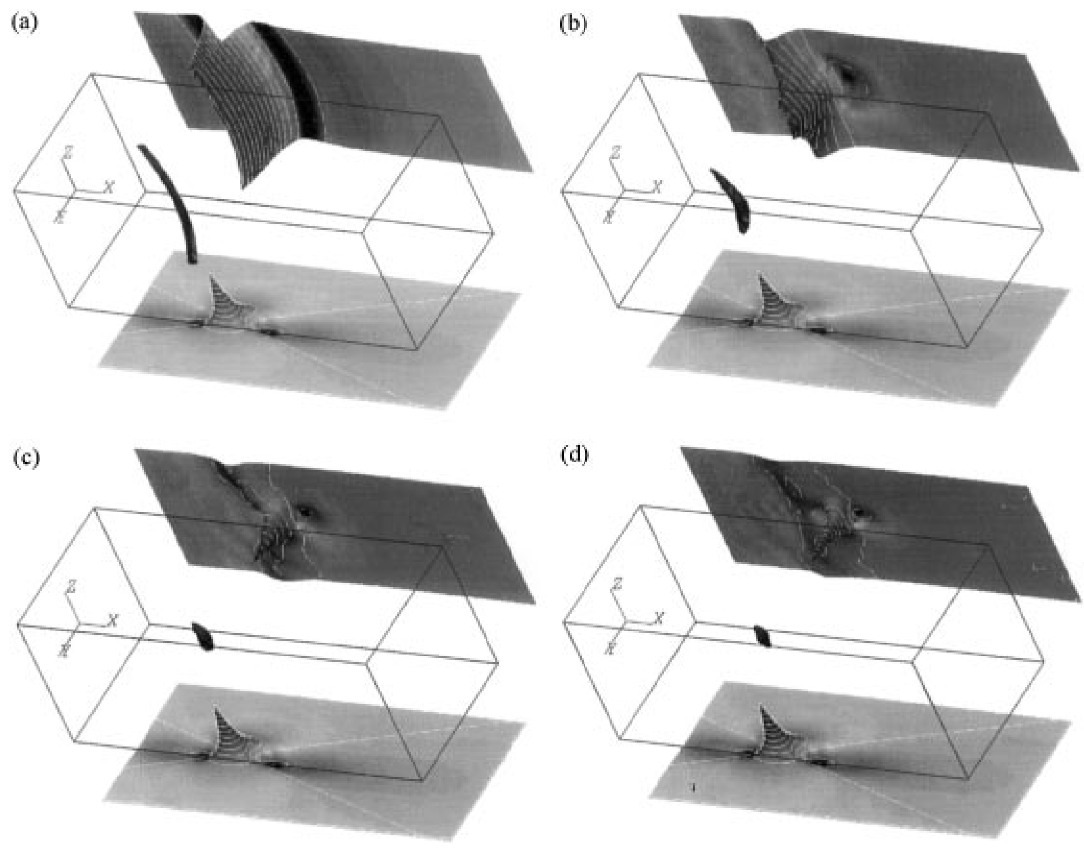

FIG. 3. As in Fig. 1, but only the innermost, $n = 8.84$, surface displayed. This figure clearly shows that a robust fragment is produced. It was obscured in Fig. 1 by the other surfaces ($n = -0.8$ and 1).

message of all the pictures is that the 2D soliton does break up when perturbed along the third direction, producing a rather robust fragment. Different constant density surfaces break up at different rates. It is obvious that the fragment is not the 3D soliton mentioned above. Apart from anything else, it lacks θ symmetry around x.

A comparison with the illustration of Ref. [14] is instructive. The fact that our 3D soliton is unstable seems to bar it from appearing even briefly here. This is an entirely new situation in this kind of simulation, to our knowledge (for a 3D structure to exist but refuse to put in an appearance as a result of 2D soliton disintegration).

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