Rigidity and Dynamics of Random Spring Networks

M. Kellomäki, J. Åström, and J. Timonen
Department of Physics, University of Jyväskylä, P.O. Box 35, FIN-40351 Jyväskylä, Finland
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The static and dynamic elastic properties of two-dimensional random networks composed of Hookean springs are analyzed. These networks are proved to be nonrigid with respect to small deformations, and the floppy mode ratio is calculated exactly. The vibrational spectrum is shown to consist only of zero-frequency and localized modes. The exponential decay of the amplitude and velocity of the transient wave front are shown to be exactly described by a quasi-one-dimensional model of noninteracting paths of propagation. [S0031-9007(96)01268-9]

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The inherent structural randomness of many materials of practical interest, and the new phenomena which appear in irregular structures, have given rise to growing research of both the static and the dynamic phenomena in disordered systems. Theoretical studies on static elastic properties of these systems have usually been concerned with their elastic moduli [1,2] and rigidity [3,4], while those on dynamic properties have mainly dealt with mobility edge, anomalous diffusion, and other features related to localization of (classical) waves [5,6]. Diffusive transport of energy in localized systems appears as a long time behavior, but, as we shall show here, there is interesting dynamics in these systems related to decaying transient modes. We report results for the dynamics of two-dimensional random networks whose structure is that of a network formed by randomly placed and oriented straight lines. The segments of lines between any two crossing points are treated as Hookean springs. We have chosen to consider this kind of structure with random geometry since it appears to describe many aspects of fibrous materials, e.g., the geometrical [7] and mechanical [8] properties of thin polymer films or paper sheets. We have also considered random networks composed of elastic beams [9]. There was an indication in this case of a very interesting transient mode. It turns out that this mode, which describes a decaying wave propagating along effectively one-dimensional random paths, exactly explains the dynamics of the random network of Hookean springs. We shall show below that this network is a nontrivial example of networks in which the Maxwellian approximation [10] of the constraint-counting method [3,4] is exact, and that the network is nonrigid.

The random geometry of the network is constructed by placing N_f one-dimensional straight lines of length l_f on a rectangle whose area is $A = L_x L_y$ (Fig. 1). The distributions of the line centers and of the orientation of the lines are both random. The density of the network is defined as $q = N_f l_f^2 / A$, which is the average number of lines on an area l_f^2. Densities are typically expressed in terms of the percolation threshold $q_c \approx 5.71$ [11]. Lines are bonded together at their crossing points (nodes). Only one pair of lines can cross at any single point because they have zero width. As the dangling ends of lines are removed for simplicity, the coordination number of each node is two, three, or four. In this work the segments of lines between two neighboring nodes are assumed to be axially rigid, linear Hookean springs with stiffness given by an apparent Young's modulus E_f and cross section S (usually we set $S = 1$). Thus the spring constant of a line segment of length l_s is $k_s = E_f S / l_s$. The mass distribution of lines is simplified by placing equal masses M on every node and assuming the segments themselves to be massless (Fig. 1). This kind of network will be called in the following the "random spring network" (RSN). The dynamics of the network can be described by the Hamiltonian

$$H = \frac{1}{2} M \sum_i \dot{\mathbf{u}}_i^2 + \frac{1}{2} \sum_{i,j} k_{ij} (\Delta l_{ij})^2 , \qquad (1)$$

where $\mathbf{u}_i$ is the displacement of node i from its place in the undeformed network, k_{ij} is the spring constant of the segment connecting nodes i and j, and Δl_{ij} is the deviation of spring length from its unstressed value (no

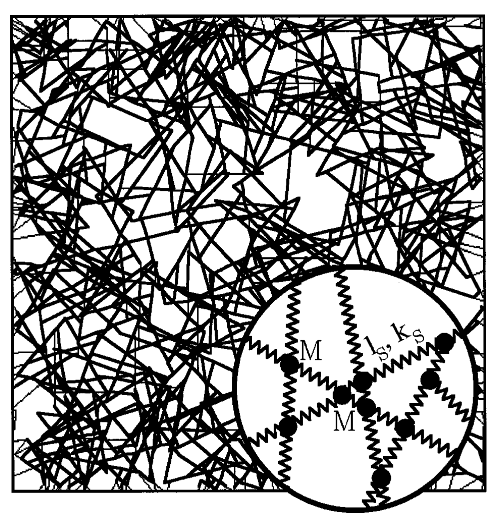

FIG. 1. A random spring network ($A = 5l_f \times 5l_f$, $q = 4q_c$). A portion of the network is magnified to show the details.

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linearization is made). The boundary conditions are such that the y direction is periodic and the right boundary ($x = L_x$) is free. The network is initially undeformed and at rest. At time $t = 0$ the x displacements of the nodes at the left boundary (at $x = 0$) are made time dependent and their y displacements are forced to be zero. Further evolution of all node displacements is calculated using the Verlet algorithm, which is widely used in classical molecular dynamics (MD) [12] and is well suited for the dynamics of the *nonlinear* Hamiltonian equation (1).

The rigidness of the RSN has not been considered before: It consists of both rigid (triangles) and nonrigid substructures (other polygons). The problem can be attacked by applying, e.g., the constraint-counting method [3,4]. The number of floppy (or zero-frequency) modes per degree of freedom f in a two-dimensional central-force system can be written as $f = 1 - \langle z \rangle / z^* + n_r$, where $\langle z \rangle$ is the average node coordination, $z^* = 4$ is the mean coordination number at the rigidity threshold, and n_r is the number of redundant bonds per degree of freedom. We can immediately deduce a couple of properties of the network. The coordination z_i of every node i is always $z_i \leq z^*$, which means that there are no overconstrained nodes. This ensures that the Maxwellian approximation [10] ($n_r = 0$) is *exact* for RSN. Using results from statistical geometry [7] we can then show that in RSN

$$f = \frac{\pi}{2q} = \frac{\pi}{2 \times 5.71} \left(\frac{q}{q_c} \right)^{-1} > 0 . \qquad (2)$$

This result suggests that RSN may not be rigid. Rigid clusters are locally composed of triangles [13] with a common side. If two triangles had a common side, there would be intersections of three different lines, which contradicts the definition of our geometrical network model. Hence in RSN triangles are always elastically isolated from each other and rigidity cannot percolate at any finite density q of the network. RSN is qualitatively similar to a diluted central-force square lattice. Both have the same $\langle z \rangle$ in the high q or perfect lattice limit, and lack a finite rigidity threshold.

We confirmed the lack of rigidity of RSN by MD-like simulations (cf. Ref. [14]) which were done by *dynamically* applying a 1% uniaxial strain in the x direction. After allowing the network to stabilize for a long time (500 times the time of flight of a wave front), the kinetic energy of the nodes was quenched by depleting the velocities of the nodes each time they passed a local potential minimum. This procedure also removed the elastic energy, thus indicating that it was stored only in the vibrations, and the final state of the network was unstressed, although it underwent a macroscopic deformation. The convergence criteria used in Refs. [1,2] could also be used in this nonlinear case even though the methods of [1,2] were not applicable. The analytically calculated floppy mode ratio [Eq. (2)] is compared in Fig. 2 with that obtained by the recently introduced topological pebble game algorithm [4,15]. The two results agree perfectly. The fact that $n_r = 0$ exactly was also directly confirmed by the pebble game.

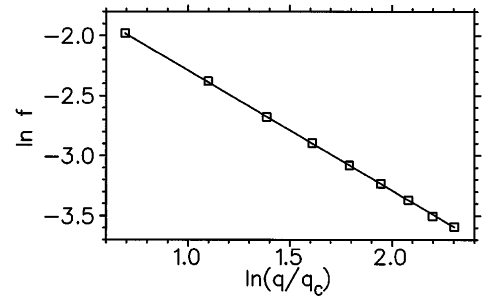

FIG. 2. Floppy mode ratio $f(q/q_c)$. The squares denote the pebble game result and the solid line is a plot of Eq. (2).

The existence of zero-frequency modes in RSN was numerically confirmed by computing the density of vibrational states as a Fourier transform of the velocity time series [16]. Moreover, the lack of acoustic modes was confirmed by computing the frequency response to a sinusoidal displacement imposed on the left boundary of the network. In Fig. 3 we show the logarithmically averaged responses $\langle g(\omega) \rangle$ for both RSN and a similar network composed of elastic beams. In both cases the parameters $q = 4q_c$, $E = 1.0$, $l_f = 1.0$, $S = 0.01$, $M = 0.01$, and two driving frequencies ω_s were used. It is evident that there is a clear response at ω_s in the random network of elastic beams, caused by driven acoustic modes, which does not appear in RSN. We can again conclude that RSN is not rigid and cannot support propagating elastic waves. All wave modes are thus floppy or localized.

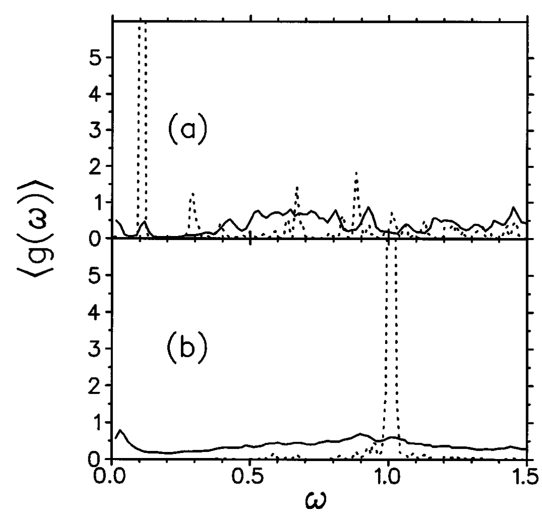

FIG. 3. Averaged frequency response $\langle g(\omega) \rangle$ for a random network composed of springs (solid line) and elastic beams (dotted line) ($q = 4q_c$, $E = 1.0$, $l_f = 1.0$, $S = 0.01$, $M = 0.01$) and two driving frequencies: (a) $\omega_s = 0.1$, (b) $\omega_s = 1.0$.

We expect that the effectively 1D transient modes observed [9] in random beam networks also exist in RSN. We looked for these modes by considering the leading edge of a semi-infinite signal generated by a longitudinal (or transverse) wave source $u(t) = A\sin(\omega_s t)$ imposed on the left boundary of the network. The leading edge or *wave front* is defined such that, at each node i, it is the first maximum of the node displacement vector $\mathbf{u}_i$. To avoid beating, we concentrated on the case in which ω_s is much larger than the eigenfrequencies of the system. The times of arrival, $t_{i,1\max}$, and the amplitudes, $A_{i,1\max} := |\mathbf{u}_i|_{1\max} = |\mathbf{u}_i(t = t_{i,1\max})|$, of the first maximum were recorded in the simulations. Thereby both the average velocity and amplitude of the wave front were determined. In the analysis the data of five networks, with approximately 37 000 nodes, were included for each set of the parameters, which means that the network area ranged from $A = 6l_f \times 150l_f$ at $q = 2q_c$ to $A = 6l_f \times 6l_f$ at $q = 10q_c$.

To better understand the observed attenuation of the wave-front amplitude, we first consider a spring chain of sawtooth geometry as a model for typical propagation paths in nonrigid two-dimensional networks. This system forms a chain with an angle 2θ between two adjacent springs of length l and masses M placed on the pivot joints. The first mass is forced to oscillate harmonically. We can consider the chain as "piecewise elastic," the amplitude of a signal being conserved along the springs, and decay being caused by the joints. For $2\theta \approx \pi/2$ the motion of a mass is not affected much by the second order force exerted by the following mass, and the amplitude at node $i + 1$ approximately satisfies $A_{i+1} = A_i \cos 2\theta$. There are $n = x/l\cos\theta$ nodes within a distance x, and in the continuum limit we end up with a decay law $A(x) = A_0 e^{-\alpha x}$, where $\alpha = l^{-1} f(\theta)$ with $f(\theta) = -\ln\cos 2\theta/\cos\theta$ in this simple case. This decay constant was checked by numerical simulations [17] on sawtooth chains. The dependence $\alpha \propto 1/l$ is evidently exact: The θ dependence was found to be qualitatively correct, and it becomes exact in the limit $2\theta \to \pi/2$ where α diverges.

In RSN the maximum length of an axially rigid subpath is the line length l_f. We can neglect the elastic diode effects [4] caused by buckling of the collinear segments because we consider only the leading wave front. Also, the dangling ends of the lines do not transport deformations. Taking this into account, the length of a typical elastic path (the bonded part of a line) is

$$l = l_f\left(1 - 2\beta\frac{\langle l_s\rangle}{l_f}\right), \tag{3}$$

where $\langle l_s\rangle = \pi l_f/2q$ [7] is the average segment (spring) length and $\beta\langle l_s\rangle$ is the effective length of the dangling end; β is a constant. The angular part $f(\theta)$ of the decay constant of sawtooth chain is replaced by an averaged quantity which is denoted by a constant α_0. This constant contains all geometrical effects of the random 2D network of straight lines, including the effective crossing angle of lines and weak couplings between the quasi-1D paths. Using Eq. (3) and $f(\theta) \equiv \alpha_0$, we find that, for $q > q_c \approx 5.71$, the average decay constant of a wave front in RSN is given by

$$\alpha_{\text{front}} = \frac{\alpha_0}{l_f(1 - \beta\pi/q)}. \tag{4}$$

This expression was found to be in excellent agreement with simulation results. In Fig. 4 (a) we show the wave-front amplitude as a function of distance for three different line lengths. The dependence $\alpha_{\text{front}} \propto 1/l_f$ is obvious, l_f being the only length scale in the problem. The density dependence of α_{front} shown in Fig. 4 (b) agrees well with Eq. (4). Within statistical errors, α_{front} is independent of all other parameters (E_f, S, M, and ω_s). We can conclude that the exponential decay arises from the *geometry* of the network. A least squares fit gave $\alpha_0 = 1.98$ and $\beta = 1.52$. The value of β indicates that the wave front travels nearly all of the available length along an individual line before transferring to a crossing line. The decay constant α_{front} approaches a nonzero value $\alpha_0 l_f^{-1}$ as $q \longrightarrow \infty$. Even in this limit the penetration depth of a signal is only $d \approx \frac{1}{2}l_f$. Notice that d is not necessarily related to the localization length [6] which refers to the long-time behavior.

We have also considered [9] elastic waves in random beam networks. In this case a signal is always divided

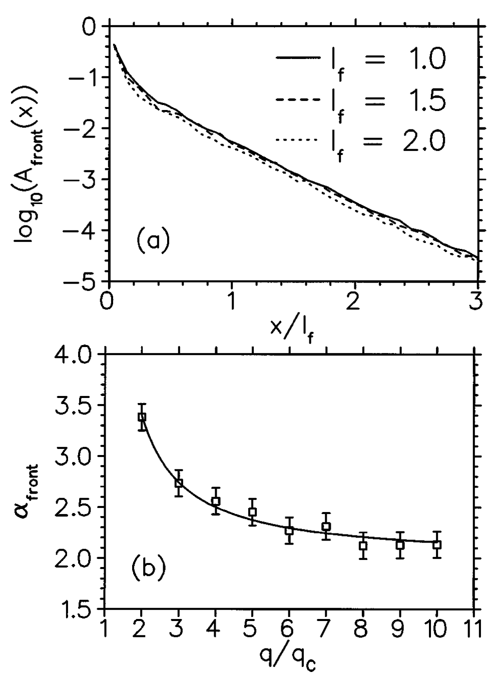

FIG. 4. (a) Amplitude decay for line lengths $l_f = 1.0$, 1.5, and 2.0. (b) Decay constant α_{front} as a function of the network density q ($l_f = 1.0$). Squares denote the simulated points and solid line is a fit by Eq. (4) ($\alpha_0 = 1.98$, $\beta = 1.52$).

into a part described by effective medium theory (acoustic modes) and another described by an ensemble of independent segment chains (transient mode). In the limit of slender beams or high frequencies, transient mode becomes more dominant. In the case of RSN, for which Young's modulus vanishes, this effectively 1D mode is the only remaining mode. For the related paths of propagation with a broken line geometry, we can express the wave-front velocity [9] in the form

$$v_{\mathrm{front}} = f_{gl}\sqrt{\frac{\pi E_f S l_f}{2qM}}\,, \tag{5}$$

where f_{gl} is a geometrical factor, and the rest of the expression is the velocity for a straight path. The factor f_{gl} reflects the influence of mainly the path geometry. Simulations of RSN were found to very accurately produce the wave-front velocity of Eq. (5). A fit to the simulated velocities gave $f_{gl} \approx 0.97$ in the density interval from $q = 3q_c$ to $q = 10q_c$. It is evident from Table I that the introduced model predicts the velocity with a relative error less than 1% for a wide range of parameters. Notice that the parameters describing the geometry of the propagation paths seem to satisfy $f_{gl} = \frac{1}{2}\alpha_0$.

In conclusion, we have analyzed the static and dynamic properties of two-dimensional central-force networks with a random geometry. The network was shown to be nonrigid at any finite density with respect to small deformations, and the number of zero-frequency modes was calculated exactly. The vibrational spectrum and the dynamic response of the network were found to be produced by nonpropagating modes only, i.e., the RSN contains only zero-frequency and localized modes. The resulting exponential decay of the wave front was found to be exactly described by a model of noninteracting quasi-one-dimensional paths of propagation. This model was also found to correctly describe the velocity of the wave front.

TABLE I. The dependence of wave-front velocity on parameters E_f, l_f, and M ($S = 1.0$). The value of the parameter is 1.0 if not explicitly given. The fitted value $f_{gl} \approx 0.97$ was used to calculate v_{front} [Eq. (5)].

Parameter	Value	v_{front}	Simulation
E_f, M, l_f	1.0	0.254	0.255
E_f	0.5	0.180	0.180
	5.0	0.568	0.573
M	0.5	0.359	0.356
	5.0	0.114	0.114
l_f	1.5	0.311	0.310
	2.0	0.359	0.361

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