Phase Transition between Coherent and Incoherent Three-Wave Interactions

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The transition from coherent to incoherent three-wave interactions with increasing bandwidth is studied. It is demonstrated analytically and numerically that this transition is sudden if the spectra of the three waves are relatively flat. The transition point, degree of coherence, and other quantities are estimated analytically and compared with numerical results. [S0031-9007(96)01280-X]

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Interactions between waves in nonlinear media have been studied for decades, notably in the contexts of plasma physics and nonlinear optics, where wave-wave interactions are involved in three- and four-wave mixing, harmonic generation, and nonlinear emission processes [1–4]. Three-wave interactions involving coalescence of two waves to produce a third, or the inverse process of decay of one wave into two others, have been among the most heavily studied. Two alternative approximations are usually made to render analysis tractable: The phase-coherent approximation in which the waves are assumed to be monochromatic and phase relationships are significant to the physics, and the random-phase approximation (RPA) in which phase is irrelevant and is averaged over [1–8]. The transition between these two key regimes is the focus of this Letter. The results are relevant both to three-wave interactions and to the wider question of nonlinear coupled oscillator systems in physics, biology, chemistry, and medicine [1–13].

For waves of bandwidth $\Delta\omega$ (defined here to be the half-width of the spectrum), a number of authors have obtained the qualitative criterion $T\Delta\omega \gg 1$ for the applicability of the RPA, where T is a characteristic nonlinear interaction time (e.g., a nonlinear growth time or period of oscillation); the reverse inequality justifies use of a phase-coherent analysis [1,2,5–8]. Subsequent work was consistent with this prediction [1,5–7]. Numerics showed that the variance of the relative phase of the modes rapidly approached its RPA value for $T\Delta\omega \gg 1$ [1]. Also, projection-operator methods enabled approximate evolution equations to be derived for moderate $\Delta\omega$ [5,7], showing a smooth decrease of coherence with increasing $\Delta\omega$. Similarly, analyses of three-wave instabilities in plasmas imply a steady reduction of coherence with increasing $\Delta\omega$, at least when one of the modes is energetically dominant [6]. However, these analyses were inconclusive as to the nature of the transition in the long-time limit: Early numerical work was restricted by the available computing power, approximations inherent in the projection technique do not permit large $\Delta\omega$ to be treated, and instability analyses did not consider long term interactions.

In this Letter, we show analytically and numerically that the breakdown of coherence of three-wave interactions can occur *suddenly* via a transition akin to first-order thermodynamic phase transitions. This sudden transition is completely unexpected and contrary to expectations based on the existing literature, which contains no hint of such behavior. Below a critical bandwidth $\Delta\omega_c$, exchange of energy between the waves remains coherent indefinitely, but the degree of coherence decreases somewhat with increasing $\Delta\omega$; for $\Delta\omega > \Delta\omega_c$, coherence decays after an initial transient. In addition to showing the existence of a transition, we calculate $\Delta\omega_c$, explore its dependence on various parameters, and study the behavior of observables as $\Delta\omega$ is varied.

Our analysis begins with the equations for the coherent decay of a wave 0 into waves 1 and 2, all with the same sign of energy (implicitly incorporating the inverse process of coalescence) [1,11]:

$$dA_0/dt = A_1A_2 \sin\Theta\,, \tag{1}$$

$$dA_{1,2}/dt = -A_0A_{2,1} \sin\Theta\,, \tag{2}$$

$$\frac{d\Theta}{dt} = \Omega - \left(\frac{A_0A_2}{A_1} + \frac{A_0A_1}{A_2} - \frac{A_1A_2}{A_0}\right)\cos\Theta\,, \tag{3}$$

where A_n and ω_n denote the real amplitude and frequency of the nth wave, $\Omega = \omega_0 - \omega_1 - \omega_2$ is the frequency mismatch, $\Theta = \theta_0 - \theta_1 - \theta_2$ in terms of the individual phases θ_n, and damping is neglected. (Note that wave dispersion links Ω and $\Delta\omega$ to the wave-number mismatch and spread, more often used in nonlinear optics.) Neutrally stable periodic solutions of (1)–(3) in terms of elliptic functions exist [1,11]; for $\Omega = 0$ these satisfy $A_0A_1A_2\cos\Theta = A_0(0)A_1(0)A_2(0)\cos\Theta(0) \equiv \Gamma$ where the argument 0 denotes an initial value.

When a finite bandwidth is introduced, one can approximate the resulting spectra by combs of N modes of amplitudes A_{nj} and frequencies ω_{nj}, with $n = 0, 1, 2$ and $j = 1, \ldots, N$. This yields the *discrete triad equations*, suitable for numerical use, in which coherent interactions between every triad of modes are followed, then summed to obtain the overall evolution of the spectra [1,5,7]:

$$\frac{dA_{0j}}{dt} = \frac{1}{N}\sum_{k'l'} A_{1k'}A_{2l'}\sin\Theta_{jk'l'}\,, \tag{4}$$

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$$\frac{d\Theta_{jkl}}{dt} = \Omega_{jkl} - \frac{1}{N}\left(\sum_{j'l'} \frac{A_{0j'}A_{2l'}}{A_{1k}} \cos\Theta_{j'kl'} + \sum_{j'k'} \frac{A_{0j'}A_{1k'}}{A_{2l}} \cos\Theta_{j'k'l} - \sum_{k'l'} \frac{A_{1k'}A_{2l'}}{A_{0j}} \cos\Theta_{jk'l'}\right), \quad (5)$$

with $j, k, l = 1, \ldots, N$, $\Omega_{jkl} = \omega_{0j} - \omega_{1k} - \omega_{2l}$, and with equations analogous to (4) for A_{1k} and A_{2l}.

Breakdown of coherence with increasing bandwidth can be studied using (4) and (5). We assume that the spectra are symmetric about central frequencies that have zero mismatch Ω_{jkl}, and measure all frequencies in a comb relative to its center in what follows. The first modes to lose coherence will be those with the largest offset frequencies ω_{nm} from the center of their comb (henceforth, the "outermost" modes). An estimate of the transition frequency at which coherence is lost can then be obtained by treating the outermost modes as test modes that interact with coherent modes having $\langle \omega_{mn} \rangle = 0$, but have no effect on them. The resulting test mode equations for a mode in comb 0 are

$$\frac{dA_{0T}}{dt} \approx \langle \Gamma/A_0 \rangle \sin\Delta\Theta_{0T}\,, \quad (6)$$

$$\frac{d\Delta\Theta_{0T}}{dt} \approx \omega_{0T} - \left\langle \frac{\Gamma}{A_0^2} \right\rangle + \left\langle \frac{\Gamma}{A_0} \right\rangle \frac{\cos\Delta\Theta_{0T}}{A_{0T}}\,, \quad (7)$$

where the subscript nT denotes a test mode in the nth comb, $\Delta\Theta_{0T}$ is the phase of the test mode relative to the mean phase of its comb, some small terms have been neglected, and a mean-field interaction with the coherent modes has been assumed (i.e., an average, denoted by $\langle \cdots \rangle$, is taken over their evolution).

Analytic solution of (6) and (7) for small oscillations shows that $\Delta\Theta_{0T}$ is bounded for $|\omega_{0T}| < \Gamma\langle A_0^{-2} \rangle \equiv \Delta\omega_{0c}$. Similar reasoning holds for $n = 1, 2$, so unbounded $\Delta\Theta_{nT}$, and consequent loss of coherence of some modes, first occurs at the offset frequency $\Delta\omega_c = \min\{\Delta\omega_{nc}\}$,

$$\Delta\omega_c \approx A_0(0)A_1(0)A_2(0)|\cos\Theta(0)| \min\{\langle A_n^{-2} \rangle\}\,, \quad (8)$$

with $n = 0, 1, 2$, and where the averages can be determined from the analytic solution of (1)–(3). This result also holds approximately for cases where some modes have large fractional oscillations in $\langle A_{nm}^2 \rangle^{1/2}$ (averaged over modes, not time), provided the largest mode is energetically dominant and has only small oscillations. In general, however, the average quantities in (8) will not give a good estimate of $\Delta\omega_c$ when all modes have large fractional oscillations in $\langle A_{nm}^2 \rangle^{1/2}$ and further analysis, including the effect of correlations, is necessary.

The above arguments yield an estimate of when the relative phase $\Delta\Theta_{nT}$ of an outermost mode first becomes unbounded. Beyond this point, the interaction of this mode with the others will be seriously weakened, and can be approximated using the RPA [1,10]. To show that coherence is then lost for the whole comb, we approximate each spectrum by a flat spectrum (i.e., with equal mode amplitudes) of half-width $\Delta\omega$ and treat the modes as test modes. If the modes start in phase, but some lose coherence with the rest of their comb, leaving fN of the N modes in the comb coherent (those with the smallest offsets ω_{nT}), the number of coherently interacting triads is fN^2. This reduces the critical frequency by a factor of f from the estimate (8) because fewer modes contribute coherently to the sums in (4) and (5) (incoherent contributions average to zero). For a flat spectrum, $f\Delta\omega_c$ is also the critical offset frequency for the outermost of the remaining fN coherent modes to lose coherence. Hence once the outermost mode of the comb decoheres, successive modes also decohere, an effect that is enhanced by the increasing noise arising from the incoherent modes. Thus for relatively flat spectra, we predict a rapid transition to a state in which no modes in the comb act coherently; near this end point, modes in the other combs also lose coherence.

An order parameter measuring the degree of coherence of three-wave interactions is the third-order correlator

$$E_c = 2\sum_{jkl} A_{0j}A_{1k}A_{2l} \cos\Theta_{jkl}\,. \quad (9)$$

For $\Delta\omega < \Delta\omega_c$, we can estimate $\langle E_c \rangle$ crudely by approximating all the modes as independent test modes of constant amplitude and averaging (9) over time and modes. For $\Theta_{ijk}(0) = 0$, this gives $\langle E_c \rangle \approx 2\prod_{n=0}^{2} A_n(0)\langle \cos(\Theta_{nT}) \rangle$. Before coherence is lost, analytic solution of (6) and (7), or their analogs for $n = 1, 2$, implies the angular amplitude ψ_n of a test mode in the nth comb is $\psi_n \approx \sin^{-1}(\omega_{nT}/\Delta\omega_{nc})$. If we average over time, assuming harmonic oscillations of the test-mode angle, and over the modes in each comb, we find

$$\frac{\langle E_c(\Delta\omega) \rangle}{\langle E_c(0) \rangle} \approx 1 - \beta \sum_{n=0}^{2} \left(\frac{\Delta\omega}{\Delta\omega_{nc}} \right)^2, \quad (10)$$

with $\beta = 1/12$; inclusion of amplitude variations and correlations in an improved analysis would change β, but not the quadratic form of (10), which is required by symmetry. For $\Delta\omega > \Delta\omega_c$, we find analytically $\langle E_c(\Delta\omega) \rangle / \langle E_c(0) \rangle \to 0$ as $N \to \infty$, assuming the terms in (9) add incoherently. We thus predict a discontinuous, first-order change in $\langle E_c \rangle$ at the transition, provided any more general analysis yields $\beta \lesssim 1/3$.

The reduction in coherence implies a reduction in the nonlinear coupling terms in (4) and (5) relative to those in (1)–(3) and, hence, an increase in the period T of the oscillations of the system, with

$$\langle E_c(\Delta\omega) \rangle T(\Delta\omega) \approx \langle E_c(0) \rangle T(0)\,. \quad (11)$$

For $\Delta\omega > \Delta\omega_c$, modes that start in phase will eventually lose coherence after a time T_b of the same order as the coherence time. For $\Delta\omega \gg \Delta\omega_c$, $T_b \sim (\Delta\omega)^{-1}$ since nonlinear effects are small. Near $\Delta\omega_c$ a critical exponent α should exist if a phase transition occurs [14], with

$$T_b \sim (\Delta\omega - \Delta\omega_c)^{-\alpha}. \quad (12)$$

To test the above predictions, we numerically solved the discrete triad equations using a Bulirsch-Stoer integrator [15] and flat initial spectra with modes all initially in phase in each comb. Analytically conserved quantities were tracked to monitor the accuracy of integration, and the output was checked against analytic results for $\Delta\omega = 0$ with and without a mismatch Ω [1,11].

Extensive sets of runs confirm the predicted existence of a sudden transition from coherence to incoherence as $\Delta\omega$ increases. Figure 1 shows a typical time series of $\langle A_{nj}^2 \rangle^{1/2}$ for $\Delta\omega > \Delta\omega_c$, $A_0(0) = 0.8$, $A_1(0) = 0.3$, and $A_2(0) = 0.1$. Initial rapid, near-periodic coherent oscillations break down after a time $T_b = 2 \pm 0.5$, after which the amplitudes fluctuate about their random-phase values [1,8]. For $\Delta\omega < \Delta\omega_c$, the coherent oscillations continue, apparently indefinitely, but reduced in amplitude relative to the $\Delta\omega = 0$ case.

Figure 2 shows numerical values of $\Delta\omega_c$ vs results from (8) for 14 sets of initial conditions with $\Theta_{jkl}(0) = 0$. The numerical results scatter around the predicted line, with small-amplitude cases always agreeing well.

Equation (8) implies $\Delta\omega_c \propto |\cos\Theta(0)|$. Figure 3 shows numerical and analytic values of $\Delta\omega_c$ vs $\cos\Theta(0)$ for $A_0(0) = 0.5$, $A_1(0) = 1$, and $A_2(0) = 2$, showing good agreement for $0.1 \lesssim \cos\Theta(0) \lesssim 1$. As $\cos\Theta(0) \to 0$, coherent oscillations become large, with Θ ranging from nearly $-\pi/2$ to nearly $\pi/2$ even for $\Delta\omega = 0$. Our averaging procedures break down in this limit and $\Delta\omega_c$ seems to exceed our estimate, possibly because fluctuations shift the system to a more stable nearby configuration.

Numerical results for $\langle E_c \rangle$, T, and $\langle E_c \rangle T$ are shown in Fig. 4 as functions of $\Delta\omega/\Delta\omega_c$ for $A_0(0) = A_1(0) = 0.9$, $A_2(0) = 2.0$, and $\Theta_{jkl}(0) = 0$, giving $\lesssim 10\%$ oscillations for $\Delta\omega = 0$. Equation (11) is accurate for $\Delta\omega \lesssim 0.5\Delta\omega_c$, with $\langle E_c \rangle T$ falling by only 10% beyond this point. The form (10) adequately approximates $\langle E_c \rangle$ and [via (11)] T, but the numerical value $\beta \approx 0.25$ is larger than the crude estimate $\beta = 1/12$ above. Beyond $\Delta\omega_c$, $\langle E_c \rangle$ is found to drop to a lower value (not shown), which decreases monotonically with N, as predicted, whereas the values shown for $\Delta\omega < \Delta\omega_c$ are almost independent of N for $N = 7 - 21$. For the system of Fig. 1, for example, $\langle E_c \rangle / \langle E_c(0) \rangle$ drops sharply from 0.67 to <0.2 for $N = 21$, consistent with the predicted first-order phase transition. For $\Delta\omega > \Delta\omega_c$, there are no stable periodic oscillations, in agreement with our theory, so T is undefined. We have investigated the effective coherence time T_b for $\Delta\omega > \Delta\omega_c$ in two different systems, finding results consistent with a phase transition with critical exponent $\alpha = 1.0 \pm 0.2$ in (12) in both cases.

The above analytical and numerical results demonstrate the existence of a previously unexpected first-order phase transition between coherent and incoherent three-wave interactions as the bandwidth is increased. Our estimates of $\Delta\omega_c$ are reasonably accurate, except for systems with large-amplitude oscillations, where correlations must be better accounted for. After multiplying by the coherent oscillation period $T \sim \pi/\max\{A_n\}$, we find a form similar to those of previous analyses (which, however, assumed a smooth transition): $T\Delta\omega_c \sim \pi\langle \prod_{n=0}^{2} [A_n/\max\{A_n\}] \rangle$. This transition is different from the smooth reduction in oscillation amplitude with increasing Ω in (3), discussed previously [1,3,11]; in that case, small coherent oscillations remain for $T\Omega \gg 1$. We argue that systems with centrally peaked spectra will tend to have higher transition frequencies than ones with flat spectra, because of their smaller typical frequency offsets. A smooth transition is even conceivable if the peak is so sharp that its core remains coherent even when the outer

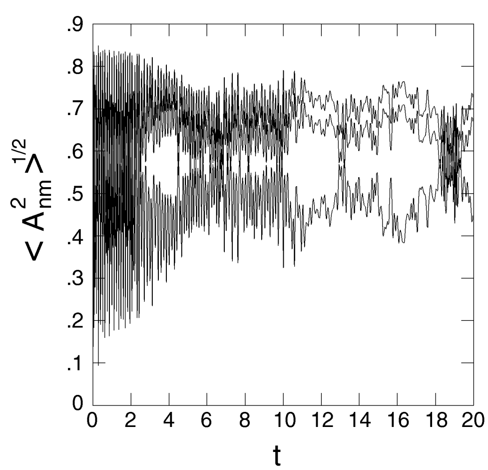

FIG. 1. Time series of $\langle A_{nm}^2 \rangle^{1/2}$ for $\Delta\omega > \Delta\omega_c$, $n = 0, 1, 2$. The curves are most easily distinguished at right.

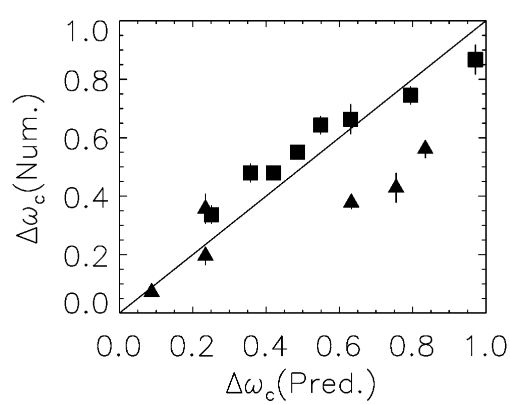

FIG. 2. Numerical vs predicted (solid line) values of $\Delta\omega_c$ for various cases with $\Theta(0) = 0$. Fractional oscillation amplitudes under 50% are indicated by squares, others by triangles.

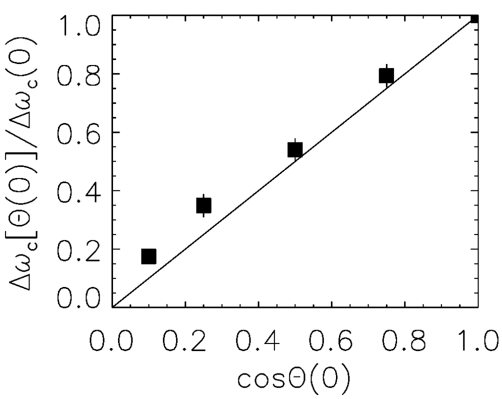

FIG. 3. Numerical values of $\Delta\omega_c[\Theta(0)]/\Delta\omega_c(0)$ vs $\cos\Theta(0)$, with all modes initially in phase in each comb. Theoretical values are given by the line.

modes decohere, a case existing in some quite different systems of coupled oscillators [10].

In practical terms, the existence of a sharp breakdown of coherence for relatively flat spectra implies that, for many semiquantitative purposes, the coherent and RPA approximations are collectively adequate to treat the entire range of bandwidths; there is no broad transition region in which neither is appropriate. Recognition of this point is important in applications of wave-wave processes in plasmas and nonlinear optics. Arguments along the lines of those in this Letter imply that N-wave interactions with $N > 3$ can also exhibit a sudden loss of coherence as their bandwidth is increased.

Nonlinear optical systems present perhaps the best opportunities to test our predictions quantitatively. For example, frequency up-conversion via three-wave mixing in a nonlinear crystal leads to periodic "sloshing" of energy between the input and output signals driven by an intense pump [3,4,11]. The present work implies that long-term sloshing will suddenly give way to random fluctuations above some critical bandwidth (or, equivalently, below a critical intensity), although there will be a coherent transient if the initial conditions are coherent. Likewise, coherent interactions between modes trapped in a cavity filled with nonlinear material (e.g., plasma or nonlinear crystal) should decohere suddenly if the bandwidth exceeds a critical value.

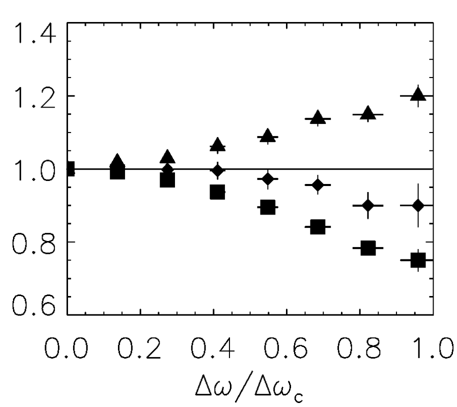

FIG. 4. Numerical values of $\langle E_c\rangle/\langle E_c(0)\rangle$ (squares), $T/T(0)$ (triangles), and $\langle E_c\rangle T/\langle E_c(0)\rangle T(0)$ (diamonds) vs $\Delta\omega/\Delta\omega_c$.

Equations (4) and (5) can be viewed as evolution equations for three distinct populations of nonlinearly coupled oscillators whose members interact in triads, one from each population. Another nonlinear oscillator system in which a phase transition has been observed, despite its different physics, is one in which every oscillator is identical and is coupled to every other one. This system was originally introduced to model coupled biological oscillators (e.g., groups of luminous insects, neurons, cardiac cells, etc.) [9,10,13]. It has been shown to have a *second-order* phase transition from coherence to incoherence. The present work shows that new types of phase transitions can occur if the oscillators are nonidentical and the form of the nonlinear interactions is modified. Other types of phase transitions have been seen recently in identical-oscillator systems, depending on the forms of the couplings and spectra [12]. This raises the possibility of similar variety here, depending on spectral structure.

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