Theorem on the Lightest Glueball State

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This paper is devoted to proving that, in QCD, the lightest glueball state must be the scalar with $J^{PC} = 0^{++}$. The proof relies upon the positivity of the path integral measure in Euclidean space and the fact that interpolating fields for all spins can be bounded by powers of the scalar glueball operator. The problem presented by the presence of vacuum condensates is circumvented by considering the time and space evolution of the propagators. [S0031-9007(96)01201-X]

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In this paper I shall show that, if glueball states exist, then the lightest one must be the 0^{++} scalar. There has recently been a renewed flurry of interest, both experimental and theoretical, in these very interesting states and the situation is beginning to clarify [1–5]. In spite of this, the situation still remains unresolved and somewhat ambiguous, so exact results such as those presented here are of some interest. Much detailed analysis has now been performed on a large amount of recent experimental data with the result that a few rather good candidates have emerged, particularly in the region 1.5–1.7 GeV [1]. Potential, bag [4] and instanton gas [2] models suggest that the lowest state should be a scalar and that its mass should be in the above range. All of these models, in spite of having the virtue of incorporating the correct low energy physics of QCD, are only effective representations of the full theory, and so their accuracy is difficult to evaluate. However, recent lattice simulations of QCD based on an extensive amount of data are in general agreement with these model results [3]. On the other hand, estimates from a field theoretic model [5] indicate that the 2^{++} tensor should be the lightest state, whereas a QCD sum rule analysis indicates that it should be the 0^{-+} pseudoscalar [6]. This disagreement between the QCD sum rules and the lattice measurements is somewhat surprising since they ought to be the least model dependent and therefore the most reliable. However, the lattice simulations use a quenched, or valence, approximation, which is not generally believed to be a major source of error, and the QCD sum rules have difficulty satisfying a low energy theorem. In any case, as already state above, the claim of this paper is that, regardless of the model or approximation used, the scalar must be the lightest glueball state. I shall now show why this must be true.

To begin, I shall first review some standard formalism as it applies to scalar and pseudoscalar glueballs before generalizing to arbitrary states. These spinless states can be described by the operators,

$$G(x) = f_G F^a_{\mu\nu}(x) F_a^{\mu\nu}(x)$$
$$\text{and} \quad \tilde{G}(x) = f_{\tilde{G}} F^a_{\mu\nu}(x) \tilde{F}_a^{\mu\nu}(x)\,, \tag{1}$$

where $\tilde{F}_a^{\mu\nu}(x) \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}(x)$ is the dual field tensor, and f_G and $f_{\tilde{G}}$ are constants. The scalar correlator

$$\Gamma(\mathbf{x}, t) \equiv \langle 0|T[G(\mathbf{x}, t)G(0)]|0\rangle \tag{2}$$

has a standard path integral representation,

$$\Gamma(\mathbf{x}, t) = \int \mathcal{D}A^a_\mu e^{i/4\int F^a_{\mu\nu}F_a^{\mu\nu}d^4x} \det(\not{D} + m) \times G(\mathbf{x}, t)G(0)\,. \tag{3}$$

A sum over quark flavors is to be understood. By inserting a complete set of states $|N\rangle$ this can also be written as

$$\Gamma(\mathbf{x}, t) = \sum_N |\langle 0|G(0)|N\rangle|^2 e^{(iE_N t - i\mathbf{p}_N\cdot\mathbf{x})}\theta(t) + (t \to -t) \tag{4}$$

from which a corresponding Kallen-Lehmann representation can be inferred (see below).

A useful subsidiary quantity to consider is (for $t > 0$)

$$Q(t) \equiv \int d^3x\, \Gamma(\mathbf{x}, t) \tag{5}$$

$$= \sum_N |\langle 0|G(0)|N\rangle|^2 \delta^{(3)}(\mathbf{p}_N) e^{iM_N t}, \tag{6}$$

where M_N is the invariant mass of the state $|N\rangle$. The Euclidean version of this (given by taking $t \to i\tau$) implies that, when $\tau \to \infty$,

$$Q_E(\tau) \equiv Q(i\tau) \approx e^{-M_0\tau}, \tag{7}$$

where M_0 is the mass of the lightest contributing state. An analogous result can be derived for $\Gamma(\mathbf{x}, t)$ via its Kallen-Lehmann representation (see below), where the exponential decay arises from the large τ or $|x|$ behavior of the free Feynman propagator. There are a couple of points worth remarking about this before proceeding. First, in pure QCD, where the $0^{\pm+}$ glueballs are expected to be the lightest states in their respective channels, $M_0 = M_G$ or $M_{\tilde{G}}$. In the full theory, however, the lightest states are those of 2 pions and 3 pions, respectively, and the glueballs become unstable resonances and mix with quark states. In that case, $M_0 = M_{2\pi}$ or $M_{3\pi}$. On the other hand, in the limit when τ becomes large, but remains

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smaller than $\sim 2M_G/\Gamma_G^2$, where Γ_G is the width of the resonance, it can be shown that the exponential decay law, Eq. (7), still remains valid but with a mass M_0 given by M_G rather than $M_{2\pi}$ (a similar result obviously also holds for the pseudoscalar case). The point is that, if there are well-defined narrow resonant states present in a particular channel, then they can be sampled by sweeping through an appropriate range of asymptotic τ values where they dominate, since τ is conjugate to M_N [7].

The basic inequality we shall employ is that, in the Euclidean region,

$$(F_{\mu\nu}^a \pm \tilde{F}_a^{\mu\nu})^2 \geq 0 \Rightarrow f_G^{-1} G_E(\mathbf{x}, \tau) \geq \pm f_{\tilde{G}}^{-1} \tilde{G}_E(\mathbf{x}, \tau), \tag{8}$$

where $G_E(\mathbf{x}, \tau) \equiv G_E(\mathbf{x}, it)$. Although this inequality holds for classical fields, it can be exploited in the quantized theory by using the path integral representation, Eq. (3), in Euclidean space where the measure is positive definite. The positivity of the measure has been skillfully used by Weingarten [8] to prove that in the quark sector the pion must be the lightest state. Here, when combined with the inequality (8), it immediately leads to the inequalities (valid for $\tau > 0$),

$$\begin{aligned} f_G^{-2} \Gamma_E(\mathbf{x}, \tau) &\geq f_{\tilde{G}}^{-2} \tilde{\Gamma}_E(\mathbf{x}, \tau) \\ \text{and} \quad f_G^{-2} Q_E(\tau) &\geq f_{\tilde{G}}^{-2} \tilde{Q}_E(\tau). \end{aligned} \tag{9}$$

By taking τ large (but $< 2M_G/\Gamma_G^2$) and using (7), the inequality

$$M_G \leq M_{\tilde{G}} \tag{10}$$

easily follows. In pure QCD where these glueballs are isolated singularities, their widths vanish and the limit $\tau \rightarrow \infty$ can be taken without constraint.

Although this is the result we want, its proof presumes the absence of a vacuum condensate $E \equiv \langle 0 | G(0) | 0 \rangle$. It is generally believed that $E \neq 0$ so the lightest state contributing to the unitarity sum in Eq. (4) is, in fact, the vacuum, in which case $M_0 = 0$ and the large τ behavior of $\Gamma_E(\mathbf{x}, \tau)$ is a constant, E^2, rather than an exponential. Thus, the inequalities (9) are trivially satisfied for asymptotic values of τ since there is no condensate in the pseudoscalar channel. To circumvent this problem it is clearly prudent to consider either the derivative of $Q(t)$ or, more generally, the time or space evolution of $\Gamma(\mathbf{x}, t)$ since these remove the offending condensate contribution. Although it will be shown below that many of the subtleties can be finessed by considering $\nabla^2 \Gamma_E(\mathbf{x}, \tau)$, it is instructive to first consider (for $\tau > 0$)

$$\dot{Q}_E(\tau) = -\sum_N |\langle 0 | G(0) | N \rangle|^2 \delta^{(3)}(\mathbf{p}_N) M_N e^{-M_N \tau}. \tag{11}$$

The vacuum state clearly does not contribute to this so its large τ behavior is, up to a factor $-M_0$, just that of Eq. (7), except that M_0 is now the mass of the lightest contributing particle state. Now (for $\tau > 0$),

$$\Gamma_E(\mathbf{x}, \tau) = \langle 0 | e^{H\tau} G_E(0) e^{-H\tau} G_E(0) | 0 \rangle, \tag{12}$$

which implies

$$\dot{\Gamma}_E(\mathbf{x}, \tau) = -\langle 0 | G_E(\mathbf{x}, \tau) H G_E(0) | 0 \rangle, \tag{13}$$

where, in the last step, the condition $H|0\rangle = 0$ has been imposed. Notice that, whereas both $Q_E(\tau)$ and $\Gamma_E(\mathbf{x}, \tau)$ are positive definite, their time derivatives are negative definite. Now, at the classical level, H is positive definite. We can therefore repeat our previous argument by working in Euclidean space and combining the inequalities (8) with a path integral representation for (13) to formally obtain (for $\tau > 0$) the inequalities,

$$\begin{aligned} f_G^{-2} \dot{\Gamma}_E(\mathbf{x}, \tau) &\leq f_{\tilde{G}}^{-2} \dot{\tilde{\Gamma}}_E(\mathbf{x}, \tau) \\ \text{and} \quad f_G^{-2} \dot{Q}_E(\tau) &\leq f_{\tilde{G}}^{-2} \dot{\tilde{Q}}_E(\tau). \end{aligned} \tag{14}$$

The large τ limit then leads to

$$f_G^{-2} M_G e^{-M_G \tau} \geq f_{\tilde{G}}^{-2} M_{\tilde{G}} e^{-M_{\tilde{G}} \tau} \tag{15}$$

from which (10) follows even in the presence of condensates.

There are some subtle points in this argument that deserve clarification, in particular, the nature of the path integral representation for (13) and the question of the vacuum energy contribution. The Hamiltonian, H, that generates time translations is *ab initio* expressed in terms of canonical momentum and coordinate field variables ($\mathbf{E}$ and $\mathbf{A}$, for example, in the axial gauge). Unfortunately, the measure of the Hamiltonian path integral in terms of these variables is not necessarily positive definite even in the Euclidean region. For the above argument to be valid, it is therefore necessary that, after integrating out the canonical momenta, the resulting measure of the Lagrangian form be positive definite and the integrand in the scalar glueball case be negative definite. In addition, we need to show that the presence of a possible vacuum energy E_0 given by $H|0\rangle = E_0|0\rangle$ [and which was defined to vanish when obtaining (13)] does not spoil the argument. To discuss these problems it is convenient to employ some of the language and results of the transfer matrix formalism used in lattice theory since this is directly formulated in the Euclidean region as a Lagrangian theory where the measure is positive definite [9].

For sufficiently small lattice spacing, a, the elements of the transfer matrix, T, taken between adjacent time slices $(n+1)a$ and na in a coordinate basis are given by

$$T_{n+1,n} = e^{-H_{n+1,n} a}. \tag{16}$$

In the pure gauge theory, $H_{n+1,n} \equiv H_{n+1,n}(\dot{\mathbf{A}}_a, \mathbf{A}_a) = 1/2(\dot{\mathbf{A}}_a^2 + \mathbf{B}_a^2)$. In this expression, $\dot{\mathbf{A}}_a \equiv [\mathbf{A}_a^{n+1} - \mathbf{A}_a^n]/a \approx \partial \mathbf{A}_a / \partial \tau$, and the color magnetic field $\mathbf{B}_a$ is understood to be derived from $\mathbf{A}_a$ in the usual way and evaluated as the average of its values at times $(n+1)a$ and na. Notice that $H_{n+1,n}$ is just the coordinate matrix element of the usual canonical Minkowski space Hamiltonian. $H(\mathbf{E}_a, \mathbf{A}_a) = 1/2(\mathbf{E}_a^2 + \mathbf{B}_a^2)$ with $\mathbf{E}_a$ replaced by $\dot{\mathbf{A}}_a$. Except for the factor f_G, this is none other than

$G_E(x)$. The extension of this to include fermions presents no difficulty and is best expressed using lattice gauge theory [9,10]. Notice that because $H_{n+1,n} \geq 0$ the transfer matrix elements satisfy the inequality: $0 \leq T_{n+1,n} \leq 1$. This bound, which follows from the existence of a Hamiltonian, plays a central role in our proof [9–11]. Up to an irrelevant normalization factor which cancels in evaluating amplitudes relative to the vacuum-vacuum amplitude, the operator form for T is $T = e^{-Ha}$, where H is to be expressed canonically in terms of conjugate variables. Thus T is basically the time evolution operator which generates the path integral,

$$\Gamma_E(\mathbf{x}, \tau) = \langle T^{N-k} G T^{k-1} G T^l \rangle . \tag{17}$$

Here $(k - l)a = \tau$ is fixed, while $Na \to \infty$ in order to pick out the vacuum expectation value. When this is expressed in the above matrix form it leads to the conventional path integral [the Euclidean version of (3)]. The positivity of the resulting measure simply reflects the positivity of $T_{n+1,n}$.

Let us apply this to $\dot{Q}_E(\tau)$ by considering the finite lattice difference

$$\begin{aligned} \Delta\Gamma_E(\mathbf{x}, \tau) &\equiv \Gamma_E(\mathbf{x}, \tau + a) - \Gamma_E(\mathbf{x}, \tau) \\ &= \langle 0 | G_E(\mathbf{x}, \tau)\,(T - 1) G_E(0) | 0 \rangle . \end{aligned} \tag{18}$$

When expressed in matrix form to generate the path integral, this immediately leads to $\Delta\Gamma_E(\mathbf{x}, \tau) \leq 0$ since $G_E(x) \geq 0$ and $T_{n+1,n} \leq 1$. Furthermore, by virtue of (8), the finite difference version of (14) can be straightforwardly derived:

$$f_G^{-2} \Delta\Gamma_E(\mathbf{x}, \tau) \leq f_{\tilde{G}}^{-2} \Delta\tilde{\Gamma}_E(\mathbf{x}, \tau) . \tag{19}$$

Finally, $\Delta Q_E(\tau)$ satisfies a representation identical to Eq. (11) but with $M_N e^{-M_N \tau}$ replaced by $(1 - e^{-M_N a}) e^{-M_N \tau}$. Thus, like (11) it is negative definite, does not receive a contribution from the vacuum condensate, and behaves the same as $e^{M_G \tau}$ for large τ. This therefore justifies the argument leading to (15) and therefore to the inequality (10).

We still need to address the question of a nonzero vacuum energy, E_0. In Eq. (11) its presence simply changes M_N to $(M_N - E_0) \geq 0$, reminding us that physical masses are to be measured relative to E_0. However, in Eq. (13), H gets replaced by $(H - E_0)$ which, inside the path integral, does not have a definite sign, thereby raising a potential problem. Consider then, instead, the lattice version of $\ddot{Q}_E(\tau)$:

$$\Delta^2 \Gamma_E(\mathbf{x}, \tau) = \langle 0 | G_E(\mathbf{x}, \tau)\,(T e^{E_0 a} - 1)^2 G_E(0) | 0 \rangle . \tag{20}$$

When expressed as a path integral in matrix form this is positive definite. Furthermore it satisfies a representation analogous to (11) but with $M_N e^{-M_N \tau}$ replaced by $[1 - e^{-(M_N - E_0)a}]^2 e^{-(M_N - E_0)\tau}$ and so behaves asymptotically as $e^{-(M_G - E_0)\tau}$. It, too, receives no contribution from the vacuum condensate. Thus, when combined with (8), this again leads to (15) and then to (10) even in the presence of a vacuum energy.

As already remarked, an alternative method for deriving the inequality is to use the space rather than the time evolution of Γ and, in particular, to consider the quantity

$$\nabla^2 \Gamma_E(\mathbf{x}, \tau) = -\langle 0 | G(\mathbf{x}, \tau) \mathbf{P}^2 G(0) | 0 \rangle , \tag{21}$$

where $\mathbf{P} = \mathbf{E}_a \times \mathbf{B}_a$ is the 3-momentum operator. Although this sidesteps some of the subtleties discussed above, the derivation requires knowledge of the asymptotic behavior of the full correlator rather than just that of $Q(\tau)$. This can be deduced from its Kallen-Lehmann representation which follows from asymptotic freedom, Eq. (4) and the fact that $G(x)$ is of dimension 4:

$$\begin{aligned} \Gamma(\mathbf{x}, t) &= E^2 + \Pi'(0) \partial^2 \delta^{(4)}(x) + \Pi(0) \delta^{(4)}(x) \\ &\quad + \partial^4 \int_{M_0^2}^{\infty} \frac{d\mu^2}{\mu^4}\, \rho(\mu^2) \Delta_F(x, \mu^2) . \end{aligned} \tag{22}$$

Here $\Pi(q^2)$ is the scalar glueball propagator [i.e., the Fourier transform of $\Gamma(x)$] whose imaginary part is the spectral weight function $\rho(q^2)$; $\Delta_F(x, \mu^2)$ is the standard free Feynman propagator. From this, one finds that the large τ behavior of $\nabla^2 \Gamma_E(\mathbf{x}, \tau)$ is, up to powers, again $e^{-M_G \tau}$. The path integral representation of (21), in which $\mathbf{E}_a$ is replaced by $\dot{\mathbf{A}}_a$ as before, is negative definite so all of the previous arguments go through leading to the inequality (10). Notice, however, that in this case the vacuum energy presents no complication since $\mathbf{P}|0\rangle \equiv 0$.

The extension of the above argument to the general case showing that the scalar must be lighter than all other glueball states can now be effected. Introduce an operator, $T_{\mu\nu\alpha\beta\ldots}(x)$, constructed out of a sufficiently long string of $F^a_{\mu\nu}(x)'s$ and $\tilde{F}^{\mu\nu}_a(x)'s$ that can, in principle, create an arbitrary physical glueball state of a given spin. Generally speaking, a given T, once constructed, can, of course, create states of many different spins, depending on the details of exactly how it is constructed. As a simple example, consider the fourth-rank tensor [12]

$$T_{\mu\nu\alpha\beta}(x) = F_{\mu\nu}(x) F_{\alpha\beta}(x) \tag{23}$$

which creates glueball states with quantum numbers 2^{++} and 0^{++}. Now, in Euclidean space, the magnitude of any component of $F^a_{\mu\nu}(x)$, or $\tilde{F}^{\mu\nu}_a(x)$, is bounded by the magnitude of $[F^a_{\mu\nu}(x) F^{\mu\nu}_a(x)]^{1/2}$. Hence, any single component of $T_{\mu\nu\alpha\beta}(x)$ must, up to a constant, be bounded by $G(x)$:

$$T_{\mu\nu\alpha\beta}(x) \leq f_G^{-1} G(x) . \tag{24}$$

This inequality is the analog of (8) and so the same line of reasoning used to exploit that inequality when proving (10) can be used here. Following the same sequence of steps leads to the conclusion that M_G must be lighter than the lightest state interpolated by $T_{\mu\nu\alpha\beta}(x)$, from which the inequality

$$M(2^{++}) \geq M(0^{++}) \equiv M_G \tag{25}$$

follows. It is worth pointing out that the pseudoscalar analog of this operator can be similarly bounded, thereby leading to the inequality $M(2^{++}) \geq M(2^{-+})$. This argument can be generalized to an arbitrary $T_{\mu\nu\alpha\beta\ldots}(x)$ since, again up to some overall constants analogous to f_G of Eq. (1), it is bounded by some power (p) of $G(x)$; i.e., for any of its components, $T_{\mu\nu\alpha\beta\ldots}(x) \leq G(x)^p$. Now, the operator $G(x)^p$ has the same quantum numbers as $G(x)$ and so can also serve as an interpolating field for the creation of the scalar glueball. The same arguments used to prove that this 0^{++} state is lighter than either the 0^{+-} or the 2^{++} can now be extended to the general case showing that it must be lighter than *any* state created by *any* T; in other words, the scalar glueball must indeed be the lightest glueball state.

Finally, we make some brief remarks about the conditions under which the bound is saturated. Clearly, the inequalities (8) become equalities when $F^a_{\mu\nu}(x) = \tilde{F}^a_{\mu\nu}(x)$ which is also the condition that minimizes the action and signals the dominance of pure nonperturbative instantons. In such a circumstance the 0^{++} and 0^{+-} will be degenerate. However, the proof of the mass inequality (10) only required (8) to be valid at asymptotic values of $|x|$. Thus, the saturation of this bound actually rests only on the weaker conditions that F be self-dual in the asymptotic region where it must vanish like a pure gauge field. Similarly, the saturation of the general inequality showing the scalar to be the lightest state occurs when *all* components of $F^a_{\mu\nu}(x)$ have the same functional dependence at asymptotic values of $|x|$. Although this is a stronger condition than required by the general asymptotic self-dual condition, it is, in fact, satisfied by the explicit single instanton solution. Thus, the splitting of the levels is determined by how much the asymptotic behavior of the nonperturbative fields differ from those of pure instantons. This therefore suggests a picture in which the overall scale of glueball masses is set by nonperturbative effects driven by instantons (which produce the confining long-range force) but that the level splittings are governed by perturbative phenomena.

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[12] For simplicity color indices, as well as the trace operator over color matrices ensuring the singlet nature of the states, have been suppressed.