

Defect Lines in the Ising Model and Boundary States on Orbifolds

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Critical phenomena in the two-dimensional Ising model with a defect line are studied using boundary conformal field theory on the $c = 1$ orbifold. Novel features of the boundary states arising from the orbifold structure, including continuously varying boundary critical exponents, are elucidated. New features of the Ising defect problem are obtained including a novel universality class of defect lines and the universal boundary to bulk crossover of the spin correlation function. [S0031-9007(96)01226-4]

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Boundary conformal field theory [1] is of considerable current interest in string theory [2], classical two-dimensional critical phenomena, and quantum impurity problems [3]. In the case of the $c = 1$ free boson, compactified onto a circle of radius r , Dirichlet (D) and Neumann (N) boundary conditions, in which the field or its dual obey $\varphi = \varphi_0$ or $\tilde{\varphi} = \tilde{\varphi}_0$, have been well studied [4]. By symmetry, the dimensions of boundary operators are independent of $\varphi_0, \tilde{\varphi}_0$. Here we extend this analysis to the $c = 1$ Z_2 orbifold [5] in which φ and $-\varphi$ are identified, showing that now the boundary dimensions depend on these parameters and that special features occur at the orbifold fixed points, $\varphi_0 = 0, \pi r, \tilde{\varphi}_0 = 0, \pi/2r$. These results are used to understand the critical behavior of a defect line in the Ising model.

We consider a two-dimensional Ising model at its critical temperature, with a (horizontal) defect line. Modifying the vertical couplings across the defect line, for a square lattice on a torus, yields the classical Hamiltonian

$$\begin{aligned} \mathcal{E} = & - \sum_{i=1,M} \sum_{j=1,N-1} [J_1 \sigma_{i,j} \sigma_{i+1,j} + J_2 \sigma_{i,j} \sigma_{i,j+1}] \\ & - \sum_{i=1,M} [J_1 \sigma_{i,N} \sigma_{i+1,N} + \tilde{J} \sigma_{i,N} \sigma_{i,1}]. \end{aligned} \quad (1)$$

Various exact results have been obtained on this model [6,7]. When the bulk couplings J_1 and J_2 are tuned to the critical point, the defect model exhibits a line of critical points depending on the details of the defect, i.e., the value of \tilde{J} in Eq. (1). The spin-spin correlation function decays along the defect line with a critical exponent which varies continuously along the line of critical points. Other types of defect lines are possible, including modification of horizontal couplings, introducing multispin couplings, imposing magnetic fields at the line, etc. Eight other universality classes can be readily obtained by cutting the system at the defect line and imposing independent boundary conditions, spin up, spin down, or free on the two sides.

To apply boundary conformal field theory (CFT) to this problem, we must “fold” the system at the defect line [8] (see Fig. 1), obtaining two Ising variables at

each point, coupled only at the boundary, a special case of the Ashkin-Teller (AT) model. The corresponding CFT is hence the $c = 1$ Z_2 orbifold [5]. We show that all known universality classes of boundary conditions in the Ising defect line problem, and some additional ones not previously known, correspond to D and N boundary conditions on the orbifold model. The universal ground-state degeneracy g [9] is calculated for all boundary conditions and the relative renormalization group stability is discussed. We have calculated the spin correlation function with the spins at arbitrary points relative to the defect line and each other for arbitrary boundary conditions; we give the result for D boundary conditions here. Details and further results will be given in a longer paper [10].

The Hilbert space for the AT model with periodic boundary conditions contains two sectors, in which the free boson obeys either periodic or twisted boundary conditions [5]. Boundary states are constructed, in general, using both sectors. (Related discussions in string theory are found in [11].) For a periodic boson, boundary states are constructed using the oscillator creation operators for right and left movers of momentum $\pm 2\pi n/\beta$, a_{Rn}^\dagger , and a_{Ln}^\dagger , where β is the length of the periodic spatial interval. In addition the zero-mode states $|w, k\rangle$ are used. Here w and k are integers which label winding numbers

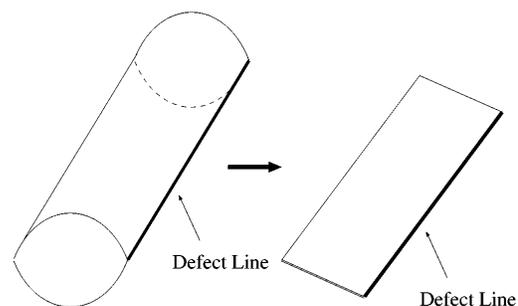


FIG. 1. The folding of the Ising model on a cylinder to a $c = 1$ theory on a strip. We fold at the defect line and also at the line on the opposite side. These lines correspond to the boundary in the folded system.

in the space and time direction, respectively. The left- and right-moving energies of these states are $(2\pi/\beta)x$ and $(2\pi/\beta)\bar{x}$ with $x = (rw + k/2r)^2$, $\bar{x} = (rw - k/2r)^2$. The Dirichlet boundary state which is an eigenstate of $\varphi(\sigma)$ with eigenvalue φ_0 is [4]

$$|D(\varphi_0)\rangle = \frac{1}{\sqrt{2r}} \exp\left[-\sum_{n=1}^{\infty} a_{Ln}^\dagger a_{Rn}^\dagger\right] \times \sum_{k=-\infty}^{\infty} e^{-ik\varphi_0/r} |(0, k)\rangle. \quad (2)$$

Orbifold model boundary states must be invariant under $\varphi \rightarrow -\varphi$; hence it is necessary to combine the D states with eigenvalue φ_0 and $-\varphi_0$:

$$|D_O(\varphi_0)\rangle = \frac{1}{\sqrt{2}} [|D(\varphi_0)\rangle + |D(-\varphi_0)\rangle]. \quad (3)$$

It is useful to consider a torus of circumferences β and $2l$ with two D defect lines at diametrically opposite locations, separated by l . After folding, this corresponds to the finite cylinder geometry considered in Ref. [1]. The partition function at inverse temperature β can be written in terms of the boundary states as

$$Z_{\varphi_0, \varphi'_0} = \langle D_O(\varphi_0) | e^{-lH_\beta^p} | D_O(\varphi'_0) \rangle = Z(\varphi_0 - \varphi'_0) + Z(\varphi_0 + \varphi'_0), \quad (4)$$

where

$$Z(\varphi_0 - \varphi'_0) = \langle D(\varphi_0) | e^{-lH_\beta^p} | D(\varphi'_0) \rangle = \frac{1}{\eta(q)} \sum_{n=-\infty}^{\infty} q^{2r^2[n+(\varphi_0-\varphi'_0)/2\pi]^2}, \quad (5)$$

with $\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ and $q = e^{-\pi\beta/l}$. $Z(\varphi_0 - \varphi'_0)$ is the partition function for a periodic boson. Note that all states in the Boltzmann sum have integer multiplicity and that, in the case $\varphi_0 = \varphi'_0$, the ground state occurs with multiplicity one. It is this physical condition which fixed the normalization of the boundary state in Eqs. (2) and (3). In the case of equal boundary conditions, the finite-size energies give the dimensions of boundary operators [1]. For a periodic boson the boundary operator spectrum can be read off from Eq. (5): $x = 2r^2n^2 + \text{integer}$. It is independent of φ_0 , as it must be by symmetry. However, this is not the case for the orbifold boson; instead, from Eq. (4), the boundary scaling dimensions vary continuously with φ_0 , corresponding to a line of fixed points.

Also note that for the orbifold case when φ_0 is at one of the fixed points, 0 or πr , the ground state occurs with multiplicity two, signaling that this is not a well-defined boundary state. To obtain well-defined Dirichlet boundary states in these two cases we must add a component from the twisted sector. Imposing the twisted boundary condition $\varphi(\sigma + \beta) = -\varphi(\sigma)$ leads to oscillators with half-integer momenta, $a_{n+1/2}$. There are no zero modes in this case and the constant part of φ must have the value 0 or πr , corresponding to two

different oscillator ground states, $|0\rangle_T$ and $|\pi r\rangle_T$. Thus we can only construct Dirichlet states in the twisted sector for these values of φ_0 . These states are

$$|D(\varphi_0)_T\rangle \equiv \exp\left[-\sum_{n=0}^{\infty} a_{L(n+1/2)}^\dagger a_{R(n+1/2)}^\dagger\right] |\varphi_0\rangle_T, \quad (6)$$

where $\varphi_0 = 0$ or πr only. There are four possible D boundary states in these cases, which give integer multiplicities when combined with the other D states and give unit multiplicity for the ground state when combined with themselves. These are

$$|D_O(\varphi_0)\pm\rangle \equiv 2^{-1/2} |D(\varphi_0)\rangle \pm 2^{-1/4} |D(\varphi_0)_T\rangle, \quad (7)$$

for $\varphi_0 = 0, \pi r$ only.

In a similar manner we can construct N states. For a periodic boson these are

$$|N(\tilde{\varphi}_0)\rangle = \sqrt{r} \sum_{w=-\infty}^{\infty} e^{-2irw\tilde{\varphi}_0} \exp\left[+\sum_{n=1}^{\infty} a_{Ln}^\dagger a_{Rn}^\dagger\right] \times |(w, 0)\rangle. \quad (8)$$

Note that in this case we use the other type of zero modes which vary in the space direction, and that the sign in the exponential for the oscillator factor is reversed. $|N(\tilde{\varphi}_0)\rangle$ is an eigenstate of the dual field, $\tilde{\varphi}(\sigma)$, of radius $1/2r$, with eigenvalue $\tilde{\varphi}_0$. We may regard $\tilde{\varphi}$ as an orbifold variable with fixed points at 0 and $\pi/2r$. Thus the orbifold N states are

$$|N_O(\tilde{\varphi}_0)\rangle = \frac{1}{\sqrt{2}} [|N(\tilde{\varphi}_0)\rangle + |N(-\tilde{\varphi}_0)\rangle], \quad (9)$$

for $0 < \tilde{\varphi}_0 < \pi/2r$. We can construct four special N states for $\tilde{\varphi}_0 = 0, \pi/2r$ by analogy with the D construction, using the twisted sector. These are constructed from the linear combinations of oscillator ground states ($|0\rangle_T \pm |\pi r\rangle_T$), respectively.

These two continuous lines, $0 < \varphi_0 < \pi r$ and $0 < \tilde{\varphi}_0 < \pi/2r$, plus 8 discrete points of boundary states obey the Ishibashi condition $[T(\sigma) - \bar{T}(\sigma)]|A\rangle = 0$, where T and \bar{T} are the left- and right-moving components of the energy-momentum tensor. All partition functions constructed from any pair of these boundary states contain only non-negative integer multiplicities. The states constructed from the same boundary state at both boundaries all have unit multiplicity for the identity operator. We conjecture that for generic values of r this is the most general set of boundary states satisfying these conditions. The analogous statement has been proven for a periodic boson [12].

We now apply the $r = 1$ case of this general analysis of orbifold boundary CFT into the Ising model with a defect line. The D boundary conditions correspond to the integrable defect line of Eq. (1). By comparing the exact spectrum [7] with that of the CFT we obtain

$$\tan(\varphi_0 - \pi/4) = \frac{\sinh[(1 - \tilde{J}/J_2)J_2/T]}{\sinh[(1 + \tilde{J}/J_2)J_2/T]} \quad (0 < \varphi < \pi), \quad (10)$$

while $\sinh(J_1/T)\sinh(J_2/T) = 1$ so that the bulk Ising model is critical. As we vary \tilde{J}/J_2 from $-\infty$ to ∞ , φ_0 decreases monotonically from $\pi/4 - \tan^{-1}[\exp(2J_2/T)]$ (between $3\pi/4$ and π) to $\pi/4 - \tan^{-1}[\exp(-2J_2/T)]$ (between 0 and $\pi/4$). Only in the extreme anisotropic limit $J_2/J_1 \rightarrow 0$, does φ_0 approach the end-point values π and 0 .

There are additional discrete universality classes of defect lines in which different boundary conditions are imposed independently on the two sides of the line. Cardy has shown [1] that there are only three universality classes of boundary conditions for an Ising model on a semi-infinite plane: spin up (\uparrow), spin down (\downarrow), and free (f). We denote the boundary state corresponding to a \uparrow boundary condition on the left of the defect and a \downarrow on the right as $|\uparrow\downarrow\rangle$, etc. The $|ff\rangle$ state simply corresponds to the Dirichlet state with $\tilde{J}/J_2 = 0$, $\varphi_0 = \pi/2$. By direct comparison of the partition functions, we have verified that the four boundary states corresponding to \uparrow and \downarrow boundary conditions are

$$|\uparrow\uparrow\rangle = |D_O(0+)\rangle, \quad |\downarrow\downarrow\rangle = |D_O(\pi+)\rangle, \quad (11)$$

with the spin-reversed states given by the corresponding “-” orbifold boundary states. It is interesting to note that the unphysical boundary state obtained by extrapolating $\varphi_0 \rightarrow 0$ without including the twisted sector can be written

$$\lim_{\varphi_0 \rightarrow 0} |D_O(\varphi_0)\rangle = |D_O(0+)\rangle + |D_O(0-)\rangle = |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle. \quad (12)$$

This limit corresponds to an infinitely strong coupling across the defect line and also a limit where the other vertical couplings $J_2/T \rightarrow 0$ while the horizontal couplings, $J_1/T \rightarrow \infty$. Thus it seems reasonable to suppose that the two horizontal chains of spins which are coupled across the defect line, (i, N) and $(i, 1)$, get locked into a perfectly ferromagnetically aligned state corresponding to $|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle$. Similarly, the limit $\varphi_0 \rightarrow \pi$ gives the perfectly aligned antiferromagnetic state.

The other four possibilities for imposing different boundary conditions on the two sides of the defect line, $|f\uparrow\rangle$, etc., correspond to the four end-point N boundary conditions in a similar way. The nature of the N line of boundary states is commented on below.

We now turn to the renormalization group stability of these various universality classes of boundary conditions. This can be addressed by checking for relevant boundary operators allowed by symmetry, and also by invoking the g theorem which states that the ground-state degeneracy g always decreases in flows from less stable to more stable fixed points. We first consider the continuous line of D fixed points, with $0 < \varphi_0 < \pi$. This has the value $g = 1$ along the entire line, indicating that this is indeed a line of stable fixed points; no flow can occur along the line. We can read off the dimensions of all boundary operators from the partition function of Eq. (4). The

only nontrivial relevant boundary operators (of dimension $x \leq 1$) have dimensions $2(\varphi_0/\pi)^2$ and $2(\varphi_0/\pi - 1)^2$. They can be shown to correspond to $\sigma_1 \pm \sigma_N$, the sum and difference of the Ising spin operators on the two sides of the defect line. If we consider a defect line with the Ising Z_2 symmetry, then the relevant boundary operators are prevented from occurring by symmetry. Thus the D boundary conditions are a stable line of fixed points which should attract generic defect lines with Z_2 symmetry. That is, we may consider arbitrary vertical and horizontal couplings near the defect line (which preserve horizontal translational symmetry) and always expect to renormalize to a D fixed point with some value of φ_0 . On the other hand, if we break the Z_2 symmetry, for example, by applying a magnetic field along the defect line, then we destabilize these fixed points. It is natural to expect that we then renormalize to one of the end-point D boundary conditions corresponding to independent spin up or down boundary conditions on the two sides of the defect line. Taking over the results for a semi-infinite Ising system, [1,9] we conclude that these boundary conditions have $g = 1/2$, and no relevant operators. They are the most stable fixed points.

The general N states have $g = \sqrt{2}$ and have relevant operators of dimension $x = 1/2$, which correspond to a product of the Ising spin operators on the two sides of the defect line. Thus, they represent unstable critical points which could be achieved only by imposing additional symmetries at the defect line [10]. We expect that adding other couplings would produce a flow away from the N boundary conditions to D boundary conditions ($g = 1$), if the Z_2 symmetry is preserved, or to fixed spin boundary conditions ($g = 1/2$) if not.

Finally, we turn to the critical spin-spin correlation function for an infinite system with a defect line of D type, for arbitrary strength of the defect. We introduce a complex spatial coordinate for the folded system, $z = x + iy$, where x, y are the two spatial coordinates; the defect is at $y = 0$ so $y \geq 0$ after folding. (If the Ising model does not have $J_1 = J_2$, square symmetry, it is necessary to rescale one of the coordinates.) The folding introduces two different spin operators σ_1 and σ_2 at each point. This universal bulk to boundary crossover function, depending in a nontrivial universal way on the cross ratio $x \equiv |(z_1 - z_2)/(z_1 - \bar{z}_2)|^2$, can be determined by standard methods in terms of the boundary state [1,13]. Using the explicit construction of the AT model conformal blocks by Zamolodchikov [14], and the D -boundary states given above, we obtain

$$\langle \sigma_j(z_1)\sigma_j(z_2) \rangle = \left(\frac{1}{4y_1y_2x} \right)^{1/8} \frac{1}{\vartheta_3(u(x))} \vartheta_3 \times (e^{2i\varphi_0}, \sqrt{u(x)}), \quad (13)$$

for two spins on the same side of the defect line (before folding). Here ϑ_j are elliptic theta functions (as defined

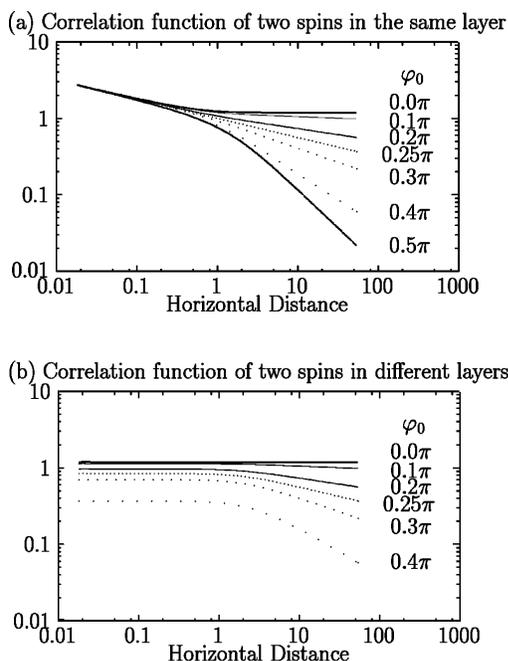


FIG. 2. The two-spin correlation for various strengths of the defect. We show the result for (a) $\langle \sigma_1 \sigma_1 \rangle$ and (b) $\langle \sigma_1 \sigma_2 \rangle$ for $\varphi_0 = 0$ (strong coupling and anisotropic limit), 0.1π , 0.2π , and 0.25π (no defect), 0.3π , 0.4π , and 0.5π (free boundary condition). They are shown as a function of the (horizontal) distance r , in a log-log plot.

in Ref. [5]) and $u(x)$ is defined by $x = [\vartheta_2(u)/\vartheta_3(u)]^4$. When the two points are on opposite sides of the defect line before folding we obtain

$$\langle \sigma_1(z_1) \sigma_2(z_2) \rangle = \left(\frac{1}{4y_1 y_2 x} \right)^{1/8} \frac{1}{\vartheta_3(u(x))} \vartheta_2 \times (e^{2i\varphi_0}, \sqrt{u(x)}). \quad (14)$$

Using MATHEMATICA, we have plotted, in Fig. 2, the correlation functions for the two points at unit distance from the boundary and separated by an arbitrary distance from each other. In the short-distance limit, $\langle \sigma_1 \sigma_1 \rangle$ converges to a unique power law, $1/4$, which is independent of the defect strength. This is the expected bulk limit. In the large-distance limit, the correlation function is governed by another exponent, $(2\varphi_0/\pi)^2$, which depends on the defect strength. This is the boundary limit. Our result interpolates between these two limits. In the “short-distance” limit, $\langle \sigma_1 \sigma_2 \rangle$ converges to a constant which depends on the defect strength. This actually corresponds to two spins

located symmetrically about the defect line. In general, $\langle \sigma_1 \sigma_2 \rangle$ is smaller than $\langle \sigma_1 \sigma_1 \rangle$ for the same defect strength and the same (horizontal) distance, as expected. Nevertheless, they asymptotically converge to the same power-law function in the large-distance limit with the constant prefactor for $\tilde{J} \neq 0$.

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Note added.—After most of this work was completed, we were informed that A.W.W. Ludwig has obtained some related results independently.

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