Comment on "Thermodynamics of Anomalous Diffusion"

In a recent Letter, Zanette and Alemany [1] have discussed the interesting point of the anomalous diffusion in the context of a generalized thermodynamics [2]. In Ref. [1] the authors claim a strong formal parallelism between the thermodynamical properties of ordinary diffusion as treated by means of the usual Boltzmann-Gibbs statistics, and those of anomalous diffusion with respect to the Tsallis statistical formulation. Even more, the authors of Ref. [1] claim to have found an explicit formula for the anomalous exponent (ν) of the mean square displacement, i.e., $\langle X(t)^2\rangle \propto t^\nu$, in terms of the q parameter which label the Tsallis thermodynamics (the particular case $q = 1$ corresponds to the standard Boltzmann-Gibbs statistics). Following Ref. [1], for the 1D case, the authors have found

$$\nu = \begin{cases} (3-q)/2 & \text{if} \quad q \le 5/3\,, \\ q-1 & \text{if} \quad q \ge 5/3\,. \end{cases}$$

However, their results are incorrect as can be inferred from the correct analysis of the Lévy-Gnedenko generalized central limit theorem [3]. For discrete times, the mean square displacement is written as $\langle X(N)^2\rangle \propto N^\nu$, where the correct ν exponent is given by

$$\nu = \begin{cases} 1 & \text{if} \quad -\infty \le q \le 5/3\,, \\ (q-1)/(3-q) & \text{if} \quad 5/3 < q < 3\,, \end{cases}$$

where the upper bound value $q < 3$ comes from normalization in 1D [3,4].

The aim of this Comment is to show what are the misleading statements of Ref. [1] which could have led to the wrong anomalous exponent ν.

First, in order to find the velocity probability distribution of a single particle $P(V)$, the authors of Ref. [1] have introduced, in a naive way, the derivative of the trajectory of a Lévy process [the mapping between q and the Lévy exponent, in d-dimension, is $q = 1 + 2/(d + \gamma)$]. We should mention that Lévy flights are defined by a jump probability $p(X)$ whose Fourier transform reads $p(k) = \exp(-a|k|^\gamma)$ (for $\gamma = 2$ an ordinary diffusive random walk is obtained [5]). Therefore, nowhere are Lévy processes differentiable (even worse, for example, the particular Cauchy process, $\gamma = 1$, has discontinuous trajectories) so one of the statements used in Ref. [1] is incorrect.

Second, in order to find the connection between the jump length (of the Lévy flights) and the energy $\epsilon = mV^2/2$ of a single particle, the authors of Ref. [1] have made use of the standard theorem of transformation of random variables to find their Eq. (2), while it is well known that this procedure of working is wrong in the context of the q-generalized statistics. For example, the fluctuation-dissipation theorem [6] and the Onsager reciprocity [4] can only be proved if q-generalized thermodynamics variables are used (this fact is analogous to having used a q-generalized theorem of transformation of random variables [4]). We want to remind readers that the whole Legendre structure of the thermodynamics will be preserved—for all values of q—only if the q-generalized intensive and extensive variables are used [7].

In order for the present comment on Ref. [1] to be complete we mention that we disagree on the conclusion about the anomalous diffusion that Zanette and Alemany have found. In order to find the time-dependent distribution, the N-jump probability distribution is given by a N-fold convolution product. Thus, using a generalized Lévy-Gnedenko central limit theorem, anomalous diffusion in 1D is only expected if $q > 5/3$. This has actually been proved in a very elegant way in Ref. [3].

Nonextensive entropy is a source of bewilderment and controversy, and it is unfortunate that misleading results like the ones given in Ref. [1] could lead to further useless controversies. This Comment is nothing but a pertinent remark on where a wrong statement can lead.

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