Elastic Instability and Curved Streamlines

Peyman Pakdel and Gareth H. McKinley
Division of Engineering and Applied Sciences, Harvard University, Cambridge, Massachusetts 02138
(Received 16 February 1996)

Hydrodynamic instabilities occur in the motion of non-Newtonian polymeric liquids at low flow rates that are entirely absent in the corresponding motions of Newtonian fluids. We develop a dimensionless criterion that characterizes the critical conditions for onset of elastic instabilities in two-dimensional, single-phase isothermal viscoelastic flows. The new dimensionless group is analogous to classical Taylor and Görtler numbers characterizing inertial instabilities of Newtonian fluids and quantifies both the curvature of the streamlines and the dynamics of the non-Newtonian fluid motion. [S0031-9007(96)01132-5]

PACS numbers: 47.20.–k, 47.50.+d, 83.50.–v

The nonlinearities in the equations of motion describing fluid flow can lead to the development of hydrodynamic instabilities stemming from the consideration of fluid inertial effects, Coriolis effects, buoyancy, surface tension, etc. [1]. In many flows involving macromolecular liquids, hydrodynamic instabilities are observed at low flow rates that are absent in the corresponding flow of Newtonian fluids [2]. In polymeric materials, the presence of a well-defined microstructure results in a complex rheological response, which in turn affects the stability of the fluid motion. Polymeric liquids exhibit significant elastic and shear-thinning phenomena [3,4] which are represented by nonlinear terms in constitutive relations that describe the state of stress in flowing polymeric materials. These viscoelastic constitutive equations are nonlinear functionals of the rate of deformation tensor, $\dot{\gamma} = (\nabla \boldsymbol{v}) + (\nabla \boldsymbol{v})^t$, where $\boldsymbol{v}(\boldsymbol{x}, t)$ is the velocity vector [4]. The complex interaction of nonlinear terms in the momentum equation and the constitutive equation give rise to a new class of unstable flows with a rich dynamical structure [5].

One of the best-documented instabilities in Newtonian flows is the Taylor-Couette instability [6]. In the Couette device, fluid is placed between two concentric cylinders and flow is generated by differentially rotating the boundaries. The Taylor instability is characterized by the generation of streamwise vorticity and development of a steady secondary cellular structure in the axial direction, known as Taylor cells [7]. The source of this instability is inertial motion of material elements along curved streamlines in which the centrifugal forces act to push the fluid outside its circular orbit [1].

A similar Taylor-Couette instability occurs in the flow of non-Newtonian fluids [5]; however, the destabilizing forces arise from nonlinear interactions between inertia, fluid shear thinning, and elasticity. Experiments with constant viscosity "ideal elastic fluids" [8] have demonstrated the presence of a purely elastic mode that occurs at negligibly small Reynolds numbers [9], and linear stability analyses with simple constitutive relations are able to predict this instability [10]. Elastic instabilities also occur in more complex geometries [11,12] that are not amenable to classical linear stability analyses due to the difficulties in obtaining analytical expressions or accurate numerical solutions for their base flows. In a two-dimensional flow field we can characterize the relative magnitude of the physical length scales in each principal direction, say, L and H, by the aspect ratio, $\Lambda = H/L$. A common characteristic of the observed elastic instabilities in such geometries is the sensitivity of the critical onset conditions and resulting spatiotemporal dynamics to the relevant geometric aspect ratio Λ.

From a micromechanical viewpoint, the motion of a polymer chain along a curvilinear streamline can be represented schematically as shown in Fig. 1. If, as a result of a radial perturbation, the polymer molecule does not lie along a streamline, the shearing motion stretches the chain nonuniformly, which in turn amplifies the non-Newtonian "hoop stress," i.e., the $\theta\theta$ component of the polymer contribution to the stress tensor $\tau_{\theta\theta}$. The nonlinear convective terms, e.g., $\lambda \boldsymbol{v} \cdot \nabla \boldsymbol{\tau}$ and $\lambda \nabla \boldsymbol{v} \cdot \boldsymbol{\tau}$, which appear in viscoelastic constitutive relations, provide the coupling with the components of the fluid momentum equation. Here, λ is a characteristic relaxation time of the fluid [4]. This perturbation to the $\theta\theta$ component of stress enters into the radial momentum balance determining the pressure field, and this pressure disturbance ultimately generates a radial

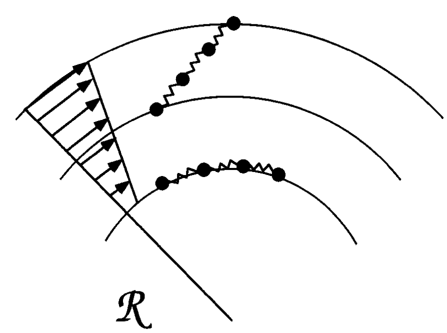

FIG. 1. Polymer chains in a curvilinear shear flow; uniform vs nonuniform chain extension.

velocity component. The presence of a radial velocity induces an axial velocity to satisfy the continuity equation, and the perturbation can thus be amplified. This complex coupling via normal stresses and curved streamlines is a distinguishing feature of elastic instabilities [10].

To characterize the onset of any instability, certain critical parameters or dimensionless groups naturally arise from dimensional analysis of the governing equation set. For the Taylor instability, the relevant dimensionless group is the Taylor number,

$$T = 2(d/r_1)\mathrm{Re}^2,$$

where r_1 is the radius of the inner cylinder, d is the gap separation, and $\mathrm{Re} = Ur_1/\nu$ is the Reynolds number. Here, U is the linear speed of the inner cylinder and ν is the kinematic viscosity. The Taylor number combines measures of fluid inertia (Re) and of the streamline radius of curvature (r_1). In elastic flows, two dimensionless groups quantifying the non-Newtonian nature of the flow may be defined; the Deborah and Weissenberg numbers [4] which are given, respectively, by

$$\mathrm{De} = \lambda/t_{\mathrm{flow}}, \qquad \mathrm{Wi} = \lambda/t_{\mathrm{def}}\,,$$

where $t_{\mathrm{flow}} \sim L/U$ is a characteristic residence time in the flow geometry, and $t_{\mathrm{def}} \sim 1/\dot{\gamma}$ is a characteristic measure of the local deformation time scale for a fluid element. In complex flow geometries, these time scales are not necessarily the same measure and characterize different physical processes [4]. By analogy to the classical Taylor number, we seek to construct a dimensionless scaling relationship for characterizing the onset of elastic instabilities with a functional dependence of the form $f(\mathcal{R}(\Lambda, t_{\mathrm{flow}}, t_{\mathrm{def}}, \lambda, \eta), \lambda, t_{\mathrm{flow}}, t_{\mathrm{def}})$. Here, $\mathcal{R}$ is a measure of streamline radius of curvature in the flow geometry and η is the fluid viscosity.

Experiments have shown that the torsional motions of elastic fluids between a rotating cone and a plate or between parallel coaxial disks are unstable to spiral disturbances [13]. Linear stability analysis of the former geometry shows that an oscillatory elastic instability appears when the following stability criterion is met [14]:

$$\mathrm{De\,Wi} = \lambda^2\omega^2/\theta_0 > 21.17\,,$$

where ω is the rate of rotation of the cone and θ_0 is the angle between the cone and the plate. In the limit of small cone angles, $\theta_0 \approx h/r$ where h is the gap separation and r is the radial distance from the apex of the cone. Therefore, we can represent the product De Wi,

$$\mathrm{De\,Wi} = \frac{\lambda r\omega}{r}\,\frac{\lambda r\omega}{h} = \frac{\lambda U}{r}\,\lambda\dot{\gamma}\,,$$

where $U = r\omega$ and $\dot{\gamma} = r\omega/h$. We define the distance $\ell \equiv \lambda U$ as *the characteristic distance over which perturbations relax along a streamline*, and $\mathcal{R} \equiv r$ as *the characteristic measure of the radius of curvature of the streamlines* in this viscoelastic base flow. The dimensionless stability criterion is then given by

$$(\mathrm{De\,Wi})_{\mathrm{crit}} = [(\ell/\mathcal{R})\,\mathrm{Wi}]_{\mathrm{crit}} = 21.17\,, \qquad (1)$$

which combines dimensionless measures of the streamline radius of curvature and of fluid elasticity. Similarly, in the linear stability analysis of the Taylor-Couette flow of ideal elastic liquids the stability boundaries vary with the parameter $(d/r_1)\,\mathrm{Wi}^2$ [10], or in our notation with the same dimensionless criterion,

$$\left(\frac{d}{r_1}\,\mathrm{Wi}^2\right)_{\mathrm{crit}} = \left[\frac{d}{r_1}\left(\frac{\lambda\omega}{d/r_1}\right)^2\right]_{\mathrm{crit}} = \left(\frac{\ell}{\mathcal{R}}\,\mathrm{Wi}\right)_{\mathrm{crit}} = 35.04\,, \qquad (2)$$

where $U = r_1\omega$ and $\mathcal{R} = r_1$ for this problem.

The flows in the above geometries are unidirectional, and the radii of curvature of the streamlines in the stable base flows are set by a single length scale. However, in a complex two-dimensional flow the local radii of curvature vary throughout the geometry as the characteristic velocity U is increased. It can be inferred from dimensional consideration of the geometry that the streamline curvature should scale with a combination of the two principal length scales of the flow, L and H. The simplest such expression is

$$1/\mathcal{R} = \alpha/L + \beta/H\,, \qquad (3)$$

where L and H are considered as length scales for the principal radii of curvature weighted by the dimensionless constants α and β. Equation (3) is applicable in both limiting cases where $\Lambda = H/L$ is either very small or large. In such extreme cases, one length scale determines the radius of curvature as in the Couette ($L = r_1$, $H \to \infty$) and cone-plate geometries ($L = r$, $H \to \infty$).

As a simple way of investigating the proposed scaling in a complex geometry, we consider the flow of two different ideal elastic fluids in a box of size $(L, H, 4L)$ where $L = 2.5$ cm, and H is adjustable in the range of $L/4$ to $4L$. The elastic fluids are composed of 0.2 and 0.35 wt.% polyisobutylene dissolved in polybutene oils, and the relaxation times determined via standard viscometric techniques [4] are $\lambda = 1.6$ and 2.5 s, respectively. The fluid is placed in the box and the top plate is translated with a constant speed in the x direction as shown schematically in Fig. 2. This "lid-driven cavity" constitutes a classical problem in fluid mechanics that has a complex dynamical structure [15–17].

In this problem, the Deborah and Weissenberg numbers are defined, respectively, by $\mathrm{De} = \lambda U/L$, $\mathrm{Wi} = \lambda U/H$. The maximum Reynolds number attained is $\mathrm{Re} \approx O(10^{-3})$, indicating that inertial effects are always negligible. At low Deborah and Weissenberg numbers, the flow is two dimensional in the x-y plane except very near the end walls. At a critical value of De and Wi, a cellular structure develops in the cavity as can be seen in

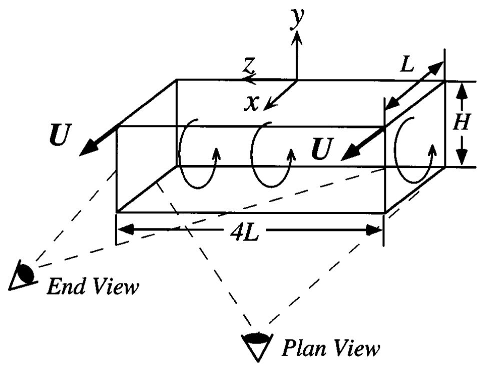

FIG. 2. The schematic diagram of the driven cavity flow; flow instability can be observed from the end view and also from below (plan view).

the photographs presented in Fig. 3. In Fig. 4, we present streak images of the flow in the x-y plane. Pronounced streamline curvature is present in the regions near the upper corners of the cavity, and as the driving velocity is increased the radius of curvature becomes smaller in the downstream corner. The onset of elastic instability is initiated in this region in the form of Taylor-Görtler-like (TGL) disturbances in the neutral or z direction [1]. The streamline curvature in the downstream corner results in a local deformation rate of $\dot{\gamma} \sim U/\mathcal{R}$, which defines the appropriate local dynamical measure of elastic effects, $\mathrm{Wi}_{\mathcal{R}} = \lambda U/\mathcal{R}$. With this definition, our dimensionless criterion [cf. Eqs. (1) and (2)] for onset of elastic instability reduces to $(\ell/\mathcal{R})^2_{\mathrm{crit}} = \mathrm{const}$, or, equivalently,

$$(\lambda U/\mathcal{R})_{\mathrm{crit}} = M_{\mathrm{crit}}\,, \tag{4}$$

where M_{crit} is a constant value for any particular fluid and is modulated by variations in other material parameters (e.g., solvent viscosity). The constants α and β in Eq. (3) are unknown; however, combining Eqs. (3) and (4), we find

$$H/\lambda U_{\mathrm{crit}} = 1/\mathrm{Wi}_{\mathrm{crit}} = \alpha\Lambda + \beta\,,$$

i.e., the stability loci measured for each cavity should lie on a straight line when $1/\mathrm{Wi}_{\mathrm{crit}}$ is plotted versus Λ. As we show in Fig. 5, this scaling argument between $1/\mathrm{Wi}_{\mathrm{crit}}$ and Λ represents our experimental observations in both fluids extremely well with a good superposition of the data. In our characterization of the radius of curvature via Eq. (3), the constants α and β do not merely provide geometrical fits to experimental data; as we show elsewhere in our analysis of more complex flow geometries, these constants scale with the rheological parameters of the fluid and the geometrical structure of the fluid motion. Furthermore, the dimensionless grouping $\ell/\mathcal{R}$ is *not* equivalent to conventional definitions of the Deborah number, and in the case of driven cavity flow it can be written in the following form:

$$\lambda U/\mathcal{R} = (\alpha + \beta/\Lambda)\,\mathrm{De}\,.$$

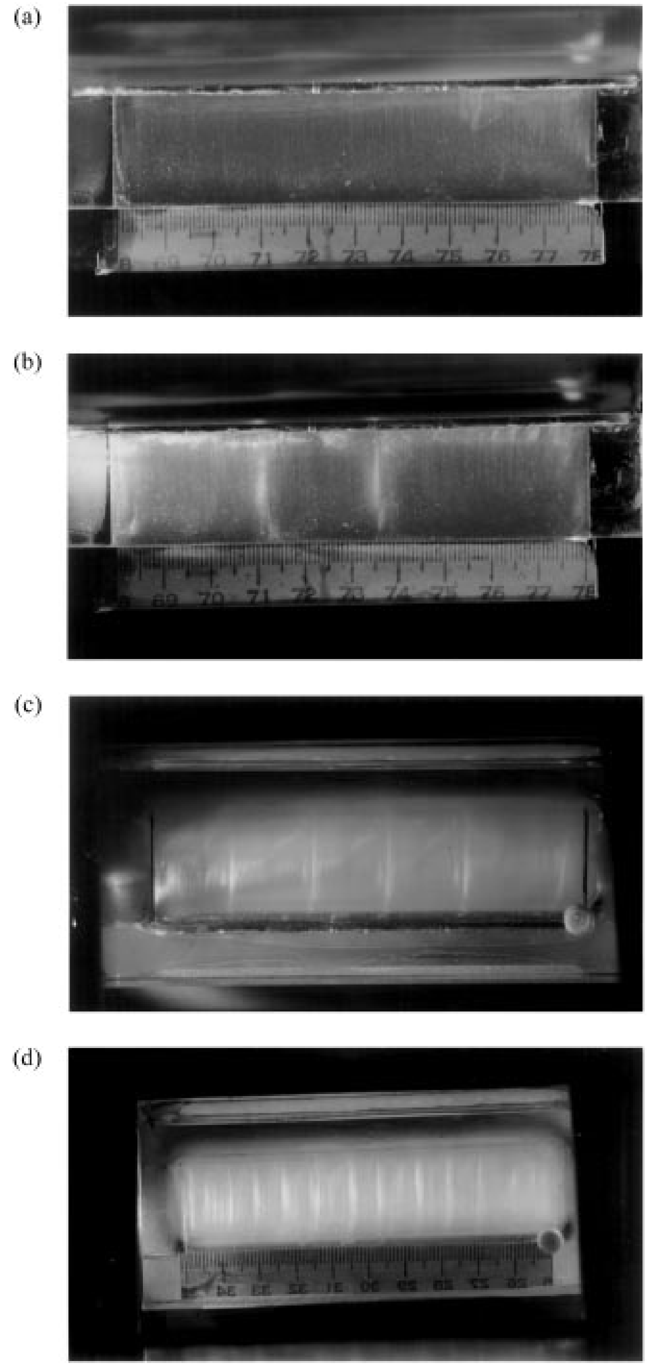

FIG. 3. Flow visualization of the elastic instability for cavities of different aspect ratios. (a) End view of the stable base flow; $\Lambda = 1$, $\mathrm{De} = 0.25$, $\mathrm{Wi} = 0.25$; (b) as the driving velocity is increased the flow in (a) becomes unstable and TGL vortices develop; $\Lambda = 1$, $\mathrm{De} = 0.35$, $\mathrm{Wi} = 0.35$. Plan views of cavities with different aspect ratios show that the spatial frequency and critical conditions vary with Λ; (c) $\Lambda = 0.5$, $\mathrm{De} = 0.29$, $\mathrm{Wi} = 0.58$; (d) $\Lambda = 0.25$, $\mathrm{De} = 0.25$, $\mathrm{Wi} = 1$.

The measure of curvature $\mathcal{R}$ is infinite for geometries that do not exhibit any curved streamlines. For example, in the rectilinear shear flow of a viscoelastic fluid ($\mathcal{R} = r_1 \rightarrow \infty$ in the Couette device) the flow is predicted to

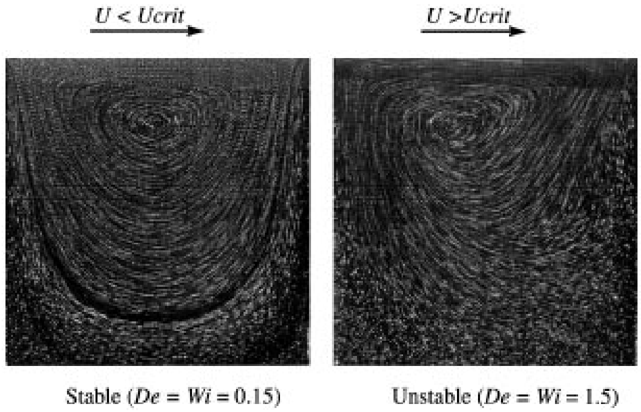

FIG. 4. Streak images of the flow field in the x-y plane for a cavity with $\Lambda = 1$: the stable versus unstable flow streamlines.

be stable to any linear disturbance at all De in agreement with theory [18].

In the case of Newtonian inertial instabilities that occur in viscous boundary layer flows over concave surfaces, the stability boundaries are characterized by the Görtler number $G^2 = (U_\infty \delta_x / \nu) \delta_x / \mathcal{R}$. Here U_∞ is the ambient undisturbed velocity and δ_x is the local thickness of the boundary layer, typically expressed as $\delta_x = (\nu x / U_\infty)^{1/2}$, where x is the streamwise distance originating from the beginning of the boundary layer [19]. Upon substitution of the expression for δ_x into the definition of G, the Görtler number reduces to a ratio of length scales, which incorporate information about the local fluid dynamics on a viscous diffusive length scale δ_x, relative to the local radius of curvature of fluid streamlines, $\mathcal{R}$. Our proposed dimensionless criterion can be thought of as the viscoelastic complement of the Görtler number. Of course, in a viscoelastic fluid there is no characteristic local boundary layer length scale, and this is replaced by the scale $\ell = \lambda U$ characterizing the relaxation of a kinematic perturbation along a viscoelastic material line. Furthermore, the destabilizing normal stress that provides the driving force in elastic instabilities does not have to increase towards the center of curvature and may, in fact, increase, decrease, or remain constant depending on the particular geometry. For example, the viscoelastic Taylor-Couette instability occurs when either the inner or outer cylinder is rotated beyond critical values. However, the inertial Taylor instability occurs only when the inner cylinder rotates faster than the outer one beyond a critical rate.

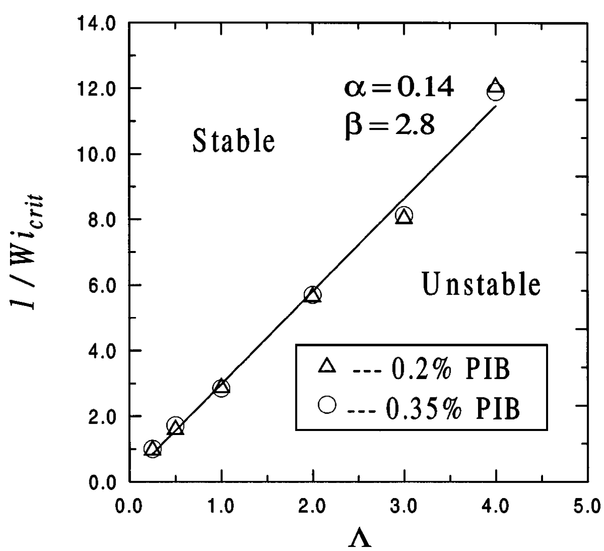

FIG. 5. The inverse of critical Weissenberg number for the onset of instability in the cavity geometry as a function of the aspect ratio: the linear relationship shows that $\ell/\mathcal{R}$ remains constant at the onset of instability. With the representation given in Eqs. (3) and (4), $\alpha = 0.14$, $\beta = 2.8$, and $M_{\mathrm{crit}} = 1.0$.

We wish to thank George M. Homsy of Stanford University and Howard A. Stone for helpful discussions. G. H. M. is grateful to the National Science Foundation for an NSF-NYI award, CTS 93-58469.

[1] P. G. Drazin and W. H. Reid, *Hydrodynamic Stability* (Cambridge University Press, Cambridge, 1981).
[2] J. R. A. Pearson, Annu. Rev. Fluid. Mech. **8**, 163 (1976).
[3] R. B. Bird and C. F. Curtiss, Phys. Today **37**, No. 1, 36 (1984).
[4] R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Liquids. Volume 1: Fluid Mechanics* (Wiley Interscience, New York, 1987), Vol. 1, 2nd ed.
[5] R. G. Larson, Rheol. Acta **31**, 213 (1992).
[6] G. I. Taylor, Philos. Trans. R. Soc. London A **223**, 289–343 (1923).
[7] P. Chossat and G. Iooss, *The Couette-Taylor Problem* (Springer, New York, 1994).
[8] D. V. Boger and Y. L. Yeow, Exp. Thermal Fluid Sci. **5**, 633 (1992).
[9] S. J. Muller, R. G. Larson, and E. S. G. Shaqfeh, Rheol. Acta **28**, 499 (1989).
[10] R. G. Larson, S. J. Muller, and E. S. G. Shaqfeh, J. Fluid Mech. **218**, 573 (1990). N.B. In this original publication the dimensionless group we define as Wi is denoted De.
[11] G. H. McKinley *et al.*, J. Fluid Mech. **223**, 441 (1991).
[12] G. H. McKinley, R. C. Armstrong, and R. A. Brown, Philos. Trans. R. Soc. London A **344**, 265 (1991).
[13] G. H. McKinley *et al.*, J. Fluid Mech. **285**, 123 (1995).
[14] D. O. Olagunju, Z. Angew. Math. Phys. **46**, 224 (1995).
[15] C. K. Aidun, N. G. Triantafillopoulos, and J. D. Benson, Phys. Fluids A **3**, 2081 (1991).
[16] C. W. Leong and J. M. Ottino, Phys. Rev. Lett. **64**, 874 (1990).
[17] F. Pan and A. Acrivos, J. Fluid Mech. **28**, 643 (1967).
[18] V. A. Gorodstov and A. I. Leonov, J. Appl. Math. Mech. **31**, 310 (1967).
[19] S. W. Saric, Annu. Rev. Fluid Mech. **26**, 379 (1994).