Walking Solitons in Quadratic Nonlinear Media

Lluis Torner, Dumitru Mazilu,* Dumitru Mihalache*
Department of Signal Theory and Communications, Polytechnic University of Catalonia, Gran Capitan UPC-D3
Barcelona, ES 08034, Spain
(Received 15 March 1996)

We study self-action of light in parametric wave interactions in nonlinear quadratic media. We show the existence of stationary solitons in the presence of Poynting vector beam walk-off or different group velocities between the waves. We discover that the new solitons constitute a two-parameter family, and they exist for different wave intensities and transverse velocities. We discuss the properties of the walking solitons and their experimental implications. [S0031-9007(96)01172-6]

PACS numbers: 42.65.Tg, 42.65.Ky, 52.35.Mw

Parametric, nonlinear, three-wave interactions play an important role in many branches of physics. They account for resonant wave mixing in media with a weak nonlinearity, quadratic in the fields, and arise in different areas of plasma physics, fluid dynamics, water and acoustic waves, electronic parametric amplifiers, and nonlinear optics [1]. In many situations of interest, the dispersive effects take place at a much longer time or space scale length than the nonlinear effects. In such cases much progress in solving the governing equations can be made by using analytical tools, including the inverse scattering transform method [2], and they have been extensively investigated for more than three decades.

However, in physical settings where the dispersive and nonlinear scale lengths are comparable, the parametric interactions of intense waves exhibit a much richer variety of phenomena than is commonly believed. The propagation of tightly focused beams or short pulses in appropriate optical media sets such a scenario, and one fascinating example of the existing new phenomena is the formation of solitons (or, more properly, solitary waves) by the mutual trapping of the interacting waves. In this paper we concentrate on parametric interactions of optical waves in quadratic nonlinear media, and we specifically study the so-called degenerate case in which a wave at a fundamental frequency interacts with its second harmonic. This case offers the additional motivation of the potential important applications of the phenomena uncovered to all-optical devices for the control of light by light.

Both (1 + 1) solitons (i.e., one transverse dimension and one propagation dimension) and higher-dimensional solitons exist in bulk crystals and in optical waveguides made of quadratic media [3–8]. Temporal solitons appear to be more difficult to form with currently available experimental conditions, but (1 + 1) and (2 + 1) bright spatial solitons have been recently observed in second harmonic generation experiments [9]. Families of zero-velocity soliton solutions of the governing equations are known to exist under ideal conditions, namely, when there is no walk-off between the interacting waves [6–8]. Temporal walk-off is due to different group velocities of the waves forming the soliton, while spatial walk-off is due to different propagation directions of energy and phase fronts in anisotropic media. Provided that nonlinear quadratic optical materials are anisotropic, beam walk-off is always present in the experiments when birefringence-tuning phase-matching techniques are used. Numerical experiments indicate that solitonlike propagation occurs in the presence of walk-off [6], but soliton solutions are not known, and a general question arises of whether stationary, non-zero-velocity solitons exist.

In this Letter we show the existence of stationary soliton solutions in the presence of walk-off between the waves. We discover that the new "walking" solitons constitute a two-parameter family, and they exist for different wave intensities and soliton velocities. The solitons do not have simple traveling-wave forms. We show the stability of physically relevant solutions and discuss their implications to the experimental excitation of solitons with different input beams.

We consider continuous wave light beams traveling in a medium with a large quadratic nonlinearity, and here we concentrate on (1 + 1) geometries. In the slowly varying envelope approximation, the beam evolution is described by the reduced normalized equations [5]

$$i\,\frac{\partial a_1}{\partial \xi} - \frac{r}{2}\,\frac{\partial^2 a_1}{\partial s^2} + a_1^* a_2 \exp(-i\beta\xi) = 0\,,$$
$$i\,\frac{\partial a_2}{\partial \xi} - \frac{\alpha}{2}\,\frac{\partial^2 a_2}{\partial s^2} - i\delta\frac{\partial a_2}{\partial s} + a_1^2 \exp(i\beta\xi) = 0\,, \quad (1)$$

where a_1 and a_2 are the amplitudes of the fundamental and second harmonic waves, and $r = -1$ for spatial solitons. The parameters α, β, and δ are given by the ratios of the coherence length ($l_c = \pi/|\Delta k|$), the diffraction lengths ($l_d = k\eta^2/2$), and the walk-off length ($l_w = \eta/\rho$). Here k is the wave vector at both frequencies, $\Delta k = 2k_1 - k_2$ is the wave-vector mismatch, ρ is the walk-off angle, and η is the beam width. One has $\alpha = -l_{d1}/l_{d2}$, $\delta = \pm 2l_{d1}/l_w$, and $\beta = \mathrm{sign}(\Delta k)\,2\pi l_{d1}/l_c$. The transverse coordinates are given in units of η, and we set for the propagation coordinate $z/l_{d1} = 2\xi$. For relevant experimental conditions, say, $l_c \sim 2.5$ mm, $\rho \sim 1°$, $\eta \sim 15\ \mu$m, one obtains $\alpha \simeq -0.5$, $\delta \sim \pm 1$, and $\beta \sim \pm 3$. For the numerics we set $\alpha = -0.5$. Equations (1) also hold for pulsed light. Then diffraction is replaced by dispersion, beam walk-off is replaced by group velocity mismatch, and r and α are given by the group velocity dispersion at the fundamental and second-harmonic frequencies.

A great insight into the properties of the new solitons we have found can be obtained from the integrals of the wave evolution. Here we shall make use of three known integrals which can be readily obtained from Noether's theorem, or directly from the governing equations, namely, the total beam power or energy flow given by the Manley-Rowe relation

$$I = I_1 + I_2 = \int \{|A_1|^2 + |A_2|^2\}\, ds\,, \qquad (2)$$

the Hamiltonian or field energy

$$\begin{aligned}\mathcal{H} = -\frac{1}{2}\int \Bigg\{ r\left|\frac{\partial A_1}{\partial s}\right|^2 + \frac{\alpha}{2}\left|\frac{\partial A_2}{\partial s}\right|^2 - \beta|A_2|^2 \\ + i\,\frac{\delta}{2}\left(A_2\frac{\partial A_2^*}{\partial s} - A_2^*\frac{\partial A_2}{\partial s}\right) \\ + (A_1^{*2}A_2 + A_1^2A_2^*)\Bigg\}\, ds\,, \qquad (3)\end{aligned}$$

and the total transverse beam momentum

$$\begin{aligned}\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2 = \frac{1}{4i}\int \Bigg\{ 2\left(A_1^*\frac{\partial A_1}{\partial s} - A_1\frac{\partial A_1^*}{\partial s}\right) \\ + \left(A_2^*\frac{\partial A_2}{\partial s} - A_2\frac{\partial A_2^*}{\partial s}\right)\Bigg\}\, ds\,, \qquad (4)\end{aligned}$$

where we have defined $A_1 = a_1$ and $A_2 = a_2\exp(-i\beta\xi)$. We shall also need the rate of power exchange between the fundamental and second harmonic waves. Writing the fields in the form $a_{1,2} = U_{1,2}\exp(i\phi_{1,2})$, where $U_{1,2}$ and $\phi_{1,2}$ are real quantities, one arrives at

$$\frac{dI_2}{d\xi} = 2\int U_1^2 U_2 \sin(\phi_2 - 2\phi_1 - \beta\xi)\, ds\,. \qquad (5)$$

We are looking for stationary solutions of Eqs. (1) describing mutually trapped beams walking off the $\xi = 0$ axis, hence we set

$$a_\nu(\xi, s) = U_\nu(\eta)\exp[i\phi_\nu(\xi,\eta)]\,, \quad \nu = 1,2\,, \qquad (6)$$

with U and ϕ being real functions, $\eta = s - v\xi$ is the transverse coordinate moving with the soliton peak, and $\phi_\nu(\xi, s) = \kappa_\nu\xi + f_\nu(\eta)$. Here v is the soliton velocity, κ_ν the nonlinear wave-number shifts induced by the wave interaction, and the functions $f_\nu(\eta)$ stand for the transverse phase fronts of the solitons. According to (5), to avoid all power exchange between the waves, one first needs $\kappa_2 = 2\kappa_1 + \beta$. Also the phase fronts should verify either $f_2(\eta) = 2f_1(\eta)$ everywhere, or alternatively $U_\nu(\eta)$ and $f_\nu(\eta)$ have to be symmetric and antisymmetric functions of the transverse coordinate η, respectively. We shall find out that only the solutions that occur in the absence of walk-off fulfill the former condition, whereas with this exception all the walking solitons fulfill the latter.

Substitution of (6) into (1) yields the coupled nonlinear ordinary differential equations that should be fulfilled by the functions $U_\nu(\eta)$ and $f_\nu(\eta)$. Namely,

$$\frac{1}{2}\ddot{U}_1 - \left[\kappa_1 - v\dot{f}_1 + \frac{1}{2}\dot{f}_1^2\right]U_1 + U_1U_2\cos(f_2 - 2f_1) = 0\,, \qquad (7)$$

$$\frac{1}{2}\ddot{f}_1U_1 + [\dot{f}_1 - v]\dot{U}_1 + U_1U_2\sin(f_2 - 2f_1) = 0\,, \qquad (8)$$

$$\frac{1}{2}\alpha\ddot{U}_2 + \left[2\kappa_1 + \beta - (v + \delta)\dot{f}_2 - \frac{1}{2}\alpha\dot{f}_2^2\right]U_2 - U_1^2\cos(f_2 - 2f_1) = 0\,, \qquad (9)$$

$$\frac{1}{2}\alpha\ddot{f}_2U_2 + \left[\alpha\dot{f}_2 + v + \delta\right]\dot{U}_2 + U_1^2\sin(f_2 - 2f_1) = 0\,, \qquad (10)$$

where the overdots indicate the derivative with respect to η. Recall that α, β, and δ are given by linear wave parameters, while the nonlinear wave-number shift κ_1 and the velocity v parametrize the sought after solutions. In the absence of walk-off one has $\delta = 0$, and zero-velocity soliton solutions of the above equations are known to exist [3–8].

Next we discuss the existence conditions of non-zero-velocity soliton solutions of the system of Eqs. (7)–(10). Walking solitons might exist for nonlinear wave-number shifts κ_1 and velocities v such that the soliton is not in resonance with linear dispersive waves. Otherwise the coupling between the waves would lead to energy leakage that would appear as Cherenkov radiation emitted from the soliton [10]. To calculate the resonance condition we define the longitudinal components of the nonlinear wave numbers of the two waves forming the soliton as $q_{\nu,\mathrm{nl}} = d\phi_\nu/d\xi$, so that $q_{\nu,\mathrm{nl}}(\eta) = \kappa_\nu - v\dot{f}_\nu(\eta)$. The values of the phase-front tilts $\dot{f}_\nu(\eta)$ far from $\eta = 0$ are given by Eqs. (8) and (10), and one has

$$\dot{f}_1(\eta \to \infty) = v\,, \quad \dot{f}_2(\eta \to \infty) = -\frac{1}{\alpha}(\delta + v)\,. \qquad (11)$$

The longitudinal wave numbers of the linear waves are given by the dispersion relations $q_{1,\mathrm{lin}} = -\frac{1}{2}\dot{f}_{1,\mathrm{lin}}^2$, $q_{2,\mathrm{lin}} = \frac{1}{2}\alpha\dot{f}_{2,\mathrm{lin}}^2 + \delta\dot{f}_{2,\mathrm{lin}}$, where $\dot{f}_{\nu,\mathrm{lin}}$ are given by (11). Cutoff occurs at the resonance conditions $q_{\nu,\mathrm{nl}}(\eta \to \infty) = q_{\nu,\mathrm{lin}}$, and we find

$$\kappa_{1,\mathrm{cut}} = \max\left\{\frac{1}{2}v^2, \frac{(\delta + v)^2}{4(-\alpha)} - \frac{\beta}{2}\right\}. \qquad (12)$$

For given values of the various involved parameters, walking solitons can exist for nonlinear wave-number shifts above these values.

The system (7)–(10) allows a trivial traveling-wave solution having the form given by (6), with the phase-front $f_\nu(\eta) = \omega_\nu\eta$. Substitution into (8) and (10) gives $\omega_1 = v$ and $\omega_2 = -(\delta + v)/\alpha$. However, this yields $f_2(\eta) = 2f_1(\eta)$, and the velocity $v = -\delta/(2\alpha + 1)$.

Because in all physically relevant situations one has $\alpha \simeq -0.5$, this expression gives a soliton velocity orders of magnitude larger than the actual velocity. Thus such solutions do not have physical relevance unless $\alpha = -0.5$ and there is no walk-off ($\delta = 0$). In such a case, the traveling-wave solution constitutes a zero-velocity transformation that is simply a consequence of the Galilean invariance of the governing equations.

To investigate the existence of stationary walking solitons, we have solved Eqs. (7)–(10) numerically using a band-matrix method to deal with the two-point boundary value problem for the four unknown functions $U_\nu(\eta)$, $f_\nu(\eta)$. We have concentrated on bright solitons. We have found that families of walking solitons exist at different values of the material parameters α, β, and δ, with different wave intensities and soliton velocities. A convenient way to represent the solutions is an energy flow-nonlinear wave number, i.e., $\kappa_1(I)$ diagram. Figure 1 shows such a diagram for different values of the wave-vector mismatch, at a representative value of the walk-off parameter. We have included the curves corresponding to the nonwalking solitons known in the absence of walk-off [8]. The properties of the families of walking soliton solutions are different in each case and a comprehensive study shall be published elsewhere. Three main points follow.

First, the shape of the walking solitons depends strongly on their velocity, similarly to solitons of other non-Galilean invariant equations (e.g., [11]). In Fig. 2 we have plotted the amplitude and the local phase-front tilt, defined as $\dot{f}_\nu(\eta)$, for two representative solitons. In some cases, the walking solitons exhibit oscillating tails with fast variations of the transverse phase front, as shown in Fig. 2(d). Second, we have verified numerically in selected cases by solving Eqs. (1) that solutions with increasing wave number for increasing energy flow are stable under propagation. This is consistent with Kolokolov's criterion applied to this case [12]. We have also found that solutions of the negatively sloped branches in the $\kappa_1(I)$ diagrams are unstable on propagation, so that they either eventually spread or they reshape, acquire a slightly different velocity, and decay into a stable walking soliton. To further confirm the robustness of the solitons, we have verified that walking solitons with different velocities are excited, e.g., with appropriately tilted inputs. The rigorous stability analysis of all the solutions remains to be done. Third, even in the presence of walk-off there are zero-velocity soliton solutions. The situation is somehow similar to an unfortunate driver who owns a car that has gotten front wheels with a default tilt towards either side of the road. In such a case, it is still possible to keep the car running straight ahead by applying an opposite turn to the steering wheel. Walking soliton solutions proceed the same way, by having the appropriate phase-front curvature. However, the excitation of such zero-velocity solitons would require an input beam that exactly matches the stationary soliton solution.

Important information about the nature and properties of the walking solitons we have found can be obtained using analytical tools, as follows. We first notice that Eqs. (1) can be written in the canonical form $i\partial A_1/\partial\xi = \delta_F \mathcal{H}/\delta_F A_1^*$, $i\partial \tilde{A}_2/\partial\xi = \delta_F \mathcal{H}/\delta_F \tilde{A}_2^*$, where δ_F stands for Fréchet or functional derivatives and $\tilde{A}_2 = A_2/\sqrt{2}$. This defines an infinite-dimensional Hamiltonian system,

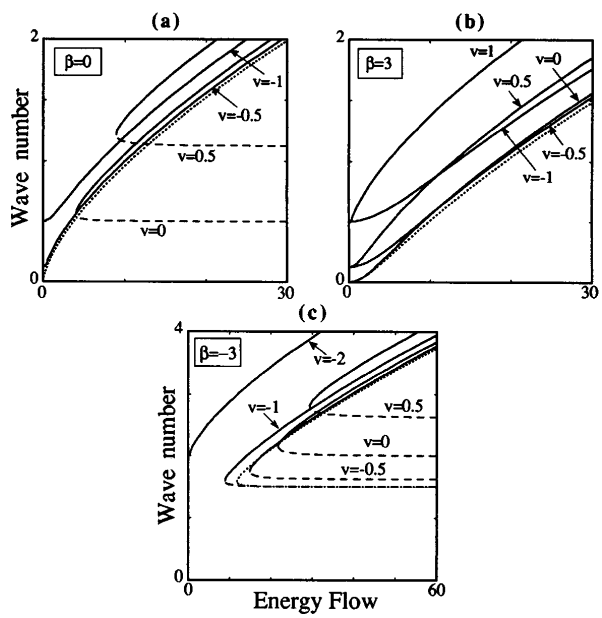

FIG. 1. Nonlinear wave number versus energy flow of the walking soliton solutions for different linear wave-vector mismatches and soliton velocities. (a) The solitons at phase matching; (b) the solitons at positive phase mismatch; (c) the solitons at negative phase mismatch. In all cases the walk-off parameter $\delta = 1$. Dashed lines: unstable solutions. Dotted lines: nonwalking solitons for $\delta = 0$.

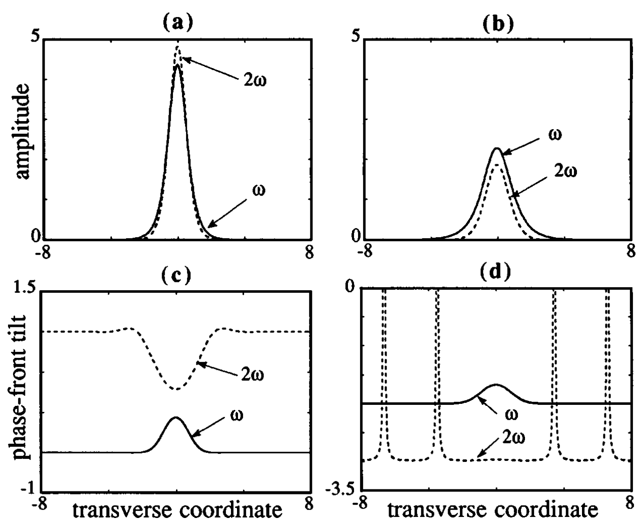

FIG. 2. Amplitude and local phase-front tilt, defined as the transverse derivative of the phase front of two walking soliton solutions as a function of the transverse coordinate. In both cases $\beta = -3$, $\delta = 1$, and $\kappa_1 = 3$. In (a) and (c) $\upsilon = -0.5$. In (b) and (d) $\upsilon = -2$.

thus it can be analyzed accordingly [13]. We first find that the stationary solutions with the form given by (6) occur at the extrema of the Hamiltonian for a given energy flow and a given transverse momentum, i.e., they occur at $\delta_F\{\mathcal{H} + \kappa_1 I - vJ\} = 0$. Now using Derrick's theorem we find that the stationary solutions are realized at

$$\mathcal{H} = -\frac{3}{5}\kappa_1 I + \frac{1}{5}\beta I_2 + \frac{4}{5}vJ - \frac{1}{5}\delta J_2. \quad (13)$$

In the absence of walk-off $\delta = 0$, and the last two terms in the right-hand side of this expression vanish for zero-velocity solutions. However, in the presence of walk-off, only the third term vanishes for zero-velocity solitons, whereas the last term contributes to the Hamiltonian. This is an important consequence of the fact that the transverse momentum of the walking solitons is not only related to their velocities, but also to their phase-front curvatures. The traveling-wave solutions that occur when $\delta = 0$ with $\alpha = -0.5$ have a flat phase front and a phase-front tilt given by the soliton velocity, hence the beam momentum is proportional to the soliton velocity. In such conditions, one finds the particlelike results $J = Iv$ and $\mathcal{H} = \mathcal{H}(v = 0) + (1/2)Iv^2$. However, Eq. (13) shows that such is not the case of the walking solitons we have discovered.

To elucidate the relation between the velocity and the momentum for the stationary walking solitons we examine the evolution of the energy centroid $\sigma(\xi) = \int s\{|a_1|^2 + |a_2|^2\}ds$. One finds $d\sigma/d\xi = J - \delta I_2 - (2\alpha + 1)J_2$. For stationary solutions of the form (6), one has $\sigma(\xi) = vI\xi$, so that the velocity of the walking solitons turns out to be

$$v = -\delta\frac{I_2}{I} + \frac{J}{I} - (2\alpha + 1)\frac{J_2}{I}. \quad (14)$$

The first term in the right-hand side of this expression has a clear physical interpretation: Walking solitons account for the mutual dragging of the waves, hence the largest the second harmonic, the highest the strength of the dragging.

Equation (14) is central to the actual excitation of walking solitons. It has to be used with caution because the dynamics of the excitation produces radiation, and such dispersive waves take energy and momentum away. Notice that in Eq. (14) the ratios I_2/I, J/I correspond to the stationary walking solitons and not to the input light. However, Eq. (14) provides a direct estimation of the velocity of the walking solitons that are excited with different inputs, as we have confirmed by performing a series of numerical experiments. In particular, Eq. (14) shows that zero-momentum inputs, i.e., having a symmetric transverse phase front, shall excite walking solitons with a velocity $v \sim -\delta I_2/I$. The ratio I_2/I depends strongly on the linear wave-vector mismatch between the waves and also on the total energy flow. In particular, at positive β the solitons have small I_2/I, and they walk slowly. At phase matching and at negative β the solitons have larger I_2/I, thus they walk faster.

In conclusion, we have found stationary, walking solitons formed by parametric wave interactions of focused beams or short pulses propagating in quadratic nonlinear optical media in the presence of spatial or temporal group velocity mismatch between the waves. The new solitons constitute a two-parameter family, and they exist for different wave intensities and transverse velocities. Our results could be also relevant to other physical settings where parametric three-wave interactions play a role in scenarios where the dispersive and nonlinear scale lengths are comparable. The approach reported can also be applied to similar problems in cubic nonlinear media. Such is the case, e.g., of highly birefringent optical fibers where the existence of walking vector solitons has been shown by Soto-Crespo *et al.* [14], by means of numerical experiments.

This work has been supported by the Spanish Government under Grant No. TIC95-1458-E. We gratefully acknowledge support by the European Union through the HCM programme. The numerical work has been carried out at the Centre de Supercomputacio de Catalunya.

*Permanent address: Department of Theoretical Physics, Institute of Atomic Physics, Bucharest, Romania.

[1] D. J. Kaup, A. Reiman, and A. Bers, Rev. Mod. Phys. **51**, 275 (1979).

[2] V. E. Zakharov and S. V. Manakov, Sov. Phys. JETP **42**, 842 (1976).

[3] Yu. N. Karamzin and A. P. Sukhorukov, Sov. Phys. JETP **41**, 414 (1976); A. A. Kanashov and A. M. Rubenchik, Physica (Amsterdam) **4D**, 122 (1981).

[4] R. Schiek, J. Opt. Soc. Am. B **10**, 1848 (1993); M. J. Werner and P. D. Drummond, J. Opt. Soc. Am. B **10**, 2390 (1993); K. Hayata and M. Koshiba, Phys. Rev. Lett. **71**, 3275 (1993).

[5] C. R. Menyuk, R. Schiek, and L. Torner, J. Opt. Soc. Am. B **11**, 2434 (1994).

[6] L. Torner, C. R. Menyuk, and G. I. Stegeman, Opt. Lett. **19**, 1615 (1994); **20**, 13 (1995); L. Torner, Opt. Commun. **114**, 136 (1995).

[7] A. V. Buryak and Y. S. Kivshar, Opt. Lett. **19**, 1612 (1994); Phys. Lett. A **197**, 407 (1995); A. V. Buryak, Y. S. Kivshar, and V. V. Steblina, Phys. Rev. A **52**, 1670 (1995).

[8] L. Torner *et al.*, Opt. Lett. **20**, 2183 (1995); Opt. Commun. **121**, 149 (1995).

[9] W. E. Torruellas *et al.*, Phys. Rev. Lett. **74**, 5036 (1995); R. Schiek, Y. Baek, and G. I. Stegeman, Phys. Rev. E **53**, 1138 (1996).

[10] N. Akhmediev and M. Karlsson, Phys. Rev. A **51**, 2602 (1995).

[11] A. B. Aceves and S. Wabnitz, Phys. Lett. A **141**, 37 (1989).

[12] A. A. Kolokolov, Lett. Nuovo Cimento **8**, 197 (1973).

[13] E. A. Kuznetsov, A. M. Rubenchik, and V. E. Zakharov, Phys. Rep. **142**, 103 (1986).

[14] J. M. Soto-Crespo, N. Akhmediev, and A. Ankiewicz, Phys. Rev. E **51**, 3547 (1995).