# Quarkonium Decay Matrix Elements from Quenched Lattice QCD

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We calculate the long-distance matrix elements for the decays of the lowest-lying *S*- and *P*-wave states of charmonium and bottomonium in quenched lattice QCD, using a nonrelativistic formulation for the heavy quarks. (The short-distance coefficients are known from perturbation theory.) In particular, we present the first calculation from QCD first principles of the color-octet contribution to *P*-wave decay—a contribution that is absent in potential models. We also give the relations between the lattice matrix elements and their continuum counterparts through one-loop order in perturbation theory. [S0031-9007(96)01086-1]

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Heavy quarkonium systems (charmonium, bottomonium) are nonrelativistic: In the CM frame, the average quark velocity $v$ satisfies $v^2 \ll 1$ ($v^2 \approx 0.3$ for charmonium and $v^2 \approx 0.1$ for bottomonium). Bodwin, Braaten, and Lepage (BBL) [1] have shown, within the framework of nonrelativistic QCD (NRQCD), that the smallness of $v^2$ allows one to express a quarkonium decay rate as a sum of terms, each of which consists of a long-distance (distance $\sim 1/M_Q v$) matrix element of a four-fermion operator in the quarkonium state multiplied by a short-distance (distance $\sim 1/M_Q$) parton-level decay rate, which may be calculated perturbatively. In particular, the decay rates for *S*-wave quarkonium through next-to-leading order in $v^2$ are given by

$$\Gamma({}^{2s+1}S_J \to X) = \mathcal{G}_1({}^{2s+1}S_J)\, 2\,\mathrm{Im}\, f_1({}^{2s+1}S_J)/M_Q^2 + \mathcal{F}_1(S)\, 2\,\mathrm{Im}\, g_1({}^{2s+1}S_J)/M_Q^4 \,. \quad (1)$$

To the lowest nontrivial order in $v^2$, the *P*-wave decay rate is given by

$$\Gamma({}^{2s+1}P_J \to X) = \mathcal{H}_1(P)\, 2\,\mathrm{Im}\, f_1({}^{2s+1}P_J)/M_Q^4 + \mathcal{H}_8(P)\, 2\,\mathrm{Im}\, f_8({}^{2s+1}S_J)/M_Q^2 \,. \quad (2)$$

The $f$'s and $g$'s are proportional to the short-distance rates for the annihilation of a $Q\bar{Q}$ pair from the indicated ${}^{2s+1}L_J$ state, while $\mathcal{G}_1$, $\mathcal{F}_1$, $\mathcal{H}_1$, and $\mathcal{H}_8$ are the long-distance matrix elements. The subscripts 1 and 8 indicate whether the $Q\bar{Q}$ pair is in a relative color-singlet or color-octet state. In this paper, we report a lattice calculation of the long-distance matrix elements in QCD for the lowest-lying *S*- and *P*-wave charmonium and bottomonium states. The calculation of $\mathcal{H}_8$ yields the first result for a heavy-quark color-octet matrix element that is based on QCD first principles.

The long-distance matrix elements are defined by

$$\mathcal{G}_1 = \langle {}^1S|\psi^\dagger \chi \chi^\dagger \psi|{}^1S\rangle \,, \quad (3a)$$

$$\mathcal{F}_1 = \langle {}^1S|\psi^\dagger \chi \psi^\dagger \left(\frac{-i}{2}\, \overleftrightarrow{\mathbf{D}}\right)^2 \chi|{}^1S\rangle \,, \quad (3b)$$

$$\mathcal{H}_1 = \langle {}^1P|\psi^\dagger \left(\frac{i}{2}\right)\overleftrightarrow{\mathbf{D}}\chi \cdot \chi^\dagger \left(\frac{i}{2}\right)\overleftrightarrow{\mathbf{D}}\psi|{}^1P\rangle \,, \quad (3c)$$

$$\mathcal{H}_8 = \langle {}^1P|\psi^\dagger T^a \chi \chi^\dagger T^a \psi|{}^1P\rangle \,. \quad (3d)$$

The terms proportional to $\mathcal{G}_1$ and $\mathcal{H}_1$ in the decay rates are those that appear in the conventional, color-singlet model [2]. In the vacuum-saturation approximation [1], which is correct up to terms of order $v^4$, $\mathcal{G}_1 = \frac{3}{2\pi}|R_S(0)|^2$ and $\mathcal{H}_1 = \frac{9}{2\pi}|R_P'(0)|^2$, where $R(0)$ is the radial wave function at the origin and $R'(0)$ is the derivative of the radial wave function at the origin. The matrix-element forms of (3) serve to define a regularized $R(0)$ and a regularized $R'(0)$ in QCD.

In contrast, the term in the *P*-wave decay rate that is proportional to $\mathcal{H}_8$ is absent in the color-singlet model. $\mathcal{H}_8$ measures the probability to find a $QQ\bar{g}$ component in *P*-wave quarkonium, with the $Q\bar{Q}$ pair in a relative *S*-wave, color-octet state. As such, it corresponds to a true field-theoretic effect of QCD that is absent in any potential model of quarkonium.

On the lattice, the long-distance matrix elements are obtained from the graphs of Fig. 1. The upper and lower graphs yield quantities that fall as $\exp[-E(|T| + |T'|)]$. The matrix element is given by the limit as $T, T' \to \infty$ of the ratio of the upper graph to the lower graph, with the same choice of sources and sinks in both graphs,

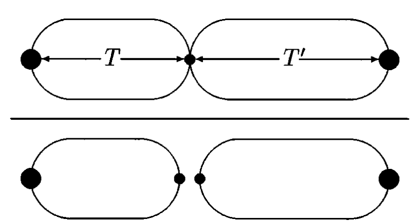


FIG. 1. Lattice calculation of matrix element of four-fermion operator. The large disks represent the sources and sinks; the smaller disks represent the four-fermion and point-source operators. The lines are the nonrelativistic quark propagators.

 

times the coefficient of the exponential falloff for the point-point quarkonium propagator. We used noisy-point and noisy-Gaussian sources and generated retarded and advanced quark propagators from each time slice. We chose the Coulomb gauge for the field configurations. This choice made implementation of extended sources simpler and allowed us to replace covariant derivatives with normal derivatives, with errors of relative order $v^2$. We calculated heavy-quark propagators $G(\mathbf{x}, t)$ on the lattice, using the nonrelativistic formulation of Lepage *et al.* [3], with an evolution equation that is valid to the lowest nontrivial order in $v^2$:

$$G(\mathbf{x}, t + 1) = (1 - H_0/2n)^n U^{\dagger}_{\mathbf{x},t} (1 - H_0/2n)^n \times G(\mathbf{x}, t) + \delta_{\mathbf{x},\mathbf{0}} \delta_{t+1,0}, \tag{4}$$

with $G(\mathbf{x}, t) = 0$ for $t < 0$, and $H_0 = -\Delta^{(2)}/2M_0 - h_0$. $\Delta^{(2)}$ is the gauge-covariant discrete Laplacian, $M_0$ the bare heavy-quark mass, and $h_0 = 3(1 - u_0)/M_0$, with $u_0 = \langle \frac{1}{3} \mathrm{Tr}\, U_{\text{plaqette}} \rangle^{1/4}$. We chose $n = 2$. Since $\mathcal{F}_1/M_Q^2$ is suppressed by $\mathcal{O}(v^2)$ relative to $\mathcal{G}_1$, it is of the same order as terms that we have neglected in the computation of $\mathcal{G}_1$. The main justification for its calculation is that there are decays, such as ${}^3S_1 \to$ light hadrons, ${}^3S_1 \to \gamma +$ light hadrons, and ${}^3S_1 \to 3\gamma$, for which the coefficient of $\mathcal{F}_1/M_Q^2$ is approximately $-5$ times that of $\mathcal{G}_1$ [4,1]. In these cases, the contributions of terms proportional to $\mathcal{F}_1$ could be important.

In our lattice calculations we used 149 quenched gauge-field configurations on a $16^3 \times 32$ lattice with $6/g^2 = 6.0$ for bottomonium and 158 configurations on an $16^3 \times 32$ lattice with $6/g^2 = 5.7$ for bottomonium and charmonium. For bottomonium we took $M_0 = 1.5$ at $6/g^2 = 6.0$ and $M_0 = 2.7$ at $6/g^2 = 5.7$. For charmonium at $6/g^2 = 5.7$ we took $M_0 = 0.69$. These values correspond to those used by the NRQCD Collaboration [5]. (Note that their mass definitions are $u_0$ times ours.) The values of $u_0$ that we used are 0.877 787 01 at $6/g^2 = 6.0$ and 0.860 826 176 0 at $6/g^2 = 5.7$. Except where we explicitly state otherwise, all quantities in this paper are in lattice units. To convert to physical units, we use inverse lattice spacings $a^{-1} = 2.4$ GeV for bottomonium at $6/g^2 = 6.0$, $a^{-1} = 1.37$ GeV for bottomonium at $6/g^2 = 5.7$, and $a^{-1} = 1.23$ GeV for charmonium at $6/g^2 = 5.7$. These are the values obtained by the NRQCD Collaboration. Our error estimates do not include the errors in these quantities.

NRQCD predicts that [1]

$$\mathcal{G}_1 / |\langle {}^1S_0 | \psi^{\dagger} \chi | 0 \rangle|^2 = 1 + \mathcal{O}(v^4), \tag{5}$$

$$\mathcal{H}_1 / \left| \langle {}^1P_1 | \psi^{\dagger} \frac{-i}{2} \overleftrightarrow{\mathbf{D}} \chi | 0 \rangle \right|^2 = 1 + \mathcal{O}(v^4), \tag{6}$$

where the vacuum-saturation approximation amounts, in this case, to ignoring the $\mathcal{O}(v^4)$ term. For bottomonium at $6/g^2 = 6.0$ we measured the $v^4$ term for $\mathcal{G}_1$ to be $1.3(1) \times 10^{-3}$. For charmonium at $6/g^2 = 5.7$ this term is approximately 1%. For $\mathcal{H}_1$, these $\mathcal{O}(v^4)$ terms, while larger than those for $\mathcal{G}_1$, are still quite small. Thus the vacuum saturation approximation is even better than one would expect. We will therefore use the vacuum-saturation values for $\mathcal{G}_1$, $\mathcal{H}_1$, and $\mathcal{F}_1$ in the discussions to follow. The lattice quantities $\mathcal{G}_{1L}$ and $\mathcal{H}_{1L}$ are then given by the coefficients of the exponentials in the $S$- and $P$-wave quarkonium propagators, respectively.

A summary of our results for the lattice matrix elements defined in (3) is presented in Table I. When a second error has been included, it is an estimate of the systematic errors associated with the parametrization of the fitting functions and with the contamination from higher states for propagators in which the separation between source and sink is too small.

To the order in $v^2$ in which we are working, our lattice matrix elements are related to their continuum counterparts by

$$\mathcal{G}_{1L} = (1 + \epsilon)\mathcal{G}_1, \tag{7a}$$

$$\mathcal{F}_{1L} = (1 + \gamma)\mathcal{F}_1 + \phi \mathcal{G}_1 \tag{7b}$$

and

$$\mathcal{H}_{1L} = (1 + \iota)\mathcal{H}_1 + \kappa \mathcal{H}_8, \tag{8a}$$

$$\mathcal{H}_{8L} = (1 + \eta)\mathcal{H}_8 + \zeta \mathcal{H}_1, \tag{8b}$$

where the subscript $L$ indicates the lattice quantity. The coefficients $\epsilon$, $\gamma$, $\phi$, $\iota$, $\eta$, and $\zeta$ are of order $\alpha_s$; $\kappa$ is of order $\alpha_s^3$. We have calculated these coefficients through order $\alpha_s$ (one loop) in tadpole-improved perturbation theory [6]. Our values for these coefficients, for minimal subtraction ($\overline{\text{MS}}$) regularization of the continuum matrix elements, are given in Table II. The accuracy of the coefficients of $\alpha_s$ in this table is estimated to be better than 1%. In computing $\zeta$, we have taken the factorization scale to be 1.3 GeV for charmonium and 4.3 GeV for bottomonium. These values correspond, approximately, to the $\overline{\text{MS}}$ heavy-quark masses. Note that $\phi$ and $\kappa$, in physical units, have dimensions of $(\text{mass})^2$, and $\zeta$ has dimensions of $1/\text{mass}^2$, whereas the other coefficients are dimensionless. If we render $\phi$, $\kappa$, and $\zeta$ dimensionless by dividing $\mathcal{F}_1$, $\phi$, $\mathcal{H}_1$ and $\kappa$ by $M_Q^2$ and by multiplying $\zeta$ by $M_Q^2$, then none of the coefficients of $\alpha_s$ is exceptionally large. Hence, the use of low-order perturbation theory appears to be reasonable.

TABLE I. Lattice decay matrix elements expressed in lattice units ($a = 1$). Note that $P$-wave bottomonium matrix elements have yet to be calculated at $6/g^2 = 5.7$.

| | Charmonium | Bottomonium | |
|---|---|---|---|
| $6/g^2$ | 5.7 | 5.7 | 6.0 |
| $\mathcal{G}_{1L}$ | 0.1317(2) (12) | 0.9156(9) (65) | 0.1489(5) (12) |
| $\mathcal{F}_{1L}(\text{non})/\mathcal{G}_{1L}$ | 1.2543(7) | 2.7456(8) | 1.3135(8) |
| $\mathcal{F}_{1L}(\text{cov})/\mathcal{G}_{1L}$ | 0.5950(5) | 2.1547(7) | 0.8522(5) |
| $\mathcal{F}_{1L}(\text{non}_2)/\mathcal{G}_{1L}$ | 0.7534(4) | 1.2205(2) | 0.7775(5) |
| $\mathcal{F}_{1L}(\text{cov}_2)/\mathcal{G}_{1L}$ | 0.5201(3) | 1.1111(2) | 0.6659(3) |
| $\mathcal{H}_{1L}$ | 0.0208(2) (20) | ⋯ | 0.0145(6) (20) |
| $\mathcal{H}_{8L}/\mathcal{H}_{1L}$ | 0.034(2) (8) | ⋯ | 0.0152(3) (20) |

TABLE II. Coefficients relating lattice and continuum matrix elements. The arguments of $\gamma$ and $\phi$ correspond to different lattice representations of $\mathcal{F}_1$. cov is a tadpole-improved [3] naive discretization of the gauge-covariant continuum operator; non is the simple, gauge-noncovariant, finite-difference operator in Coulomb gauge; the subscript 2 indicates a difference operator with spacings of two lattice units.

| | Charmonium | Bottomonium | |
|---|---|---|---|
| $6/g^2$ | 5.7 | 5.7 | 6.0 |
| $\epsilon$ | $-0.7326\alpha_s$ | $0.2983\alpha_s$ | $-0.4877\alpha_s$ |
| $\gamma$(non) | $-0.02578\alpha_s$ | $-1.248\alpha_s$ | $-0.9117\alpha_s$ |
| $\gamma$(cov) | $-2.860\alpha_s$ | $-2.192\alpha_s$ | $-2.560\alpha_s$ |
| $\gamma$(non$_2$) | $-0.2774\alpha_s$ | $-1.096\alpha_s$ | $-0.9236\alpha_s$ |
| $\phi$(non) | $1.486\alpha_s$ | $10.90\alpha_s$ | $4.418\alpha_s$ |
| $\phi$(cov) | $0.3928\alpha_s$ | $9.808\alpha_s$ | $3.325\alpha_s$ |
| $\phi$(non$_2$) | $1.004\alpha_s$ | $6.096\alpha_s$ | $2.863\alpha_s$ |
| $\iota$ | $-0.7603\alpha_s$ | $-1.852\alpha_s$ | $-1.191\alpha_s$ |
| $\eta$ | $0.09157\alpha_s$ | $-0.03728\alpha_s$ | $0.06096\alpha_s$ |
| $\zeta$ | $-0.1785\alpha_s$ | $-0.006011\alpha_s$ | $-0.01862\alpha_s$ |

For those coefficients that arise from a positive integrand, the method of Lepage and Mackenzie [6] yields an optimal scale for $\alpha_s$ that is close to $1/a$. Thus we choose $\alpha_s = \alpha_V(1/a) = 0.3552$ at $6/g^2 = 5.7$ and $0.2467$ at $6/g^2 = 6.0$.

Substituting the numerical values from Tables I and II into (7) and (8), we obtain the results shown in Table III. In Table III, the first and second errors in the lattice results are from the statistical and systematic errors in Table I. The third error is an estimate of the systematic error that arises from the neglect of terms of higher order in $\alpha_s$ in the coefficients of Table II. It is obtained by taking the uncertainty in the coefficients to be either $\alpha_s^2$ times the zeroth order term (if any) or $\alpha_s$ times the magnitude of the first order term, whichever is larger. In the case of $\mathcal{F}_1/\mathcal{G}_1$, the uncertainty is large, so we have presented our results as ranges of values.

For purposes of comparison, we have also shown in Table III the experimental (phenomenological) results, where available, for the matrix elements that we have computed. The phenomenological results for $\mathcal{G}_1$ were extracted from the measured decay rates for $J/\psi \to e^+e^-$, $\eta_c \to \gamma\gamma$, and $\Upsilon \to e^+e^-$ [7], using the expressions in Ref. [1]. Values for $\mathcal{F}_1/\mathcal{G}_1$ for $J/\psi$ are those of Ko, Lee, and Song [8]. The results for $\mathcal{H}_1$ and $\mathcal{H}_8/\mathcal{H}_1$ for $\chi_c$ are from [9]; the first error is experimental, the second theoretical. For $P$-wave bottomonium, there is as yet no published data on decays into light hadrons, photons, and/or leptons. The extraction of phenomenological matrix elements from the experimental data requires values for the heavy-quark masses. Our choices correspond to pole masses of 5.0 GeV for the $b$ quark, the result obtained by the NRQCD Collaboration [5], and 1.5 GeV for the $c$ quark [10]. We also require values for $\alpha_s$ and $\alpha$ in order to evaluate the partonic decay rates. For these we used $\alpha_s(M_c) = 0.243$, $\alpha_s(M_b) = 0.179$, $\alpha(M_c) = 1/133.3$, and $\alpha(M_b) = 1/132$.

In the above analysis, we have not taken into account the errors due to the omission of terms of higher order in $v^2$. These could be as large as 10% for bottomonium and 30% for charmonium. For $\mathcal{G}_{1L}$, the NRQCD Collaboration has published results that are accurate to the next-to-leading order in $v^2$. Since these higher-order results distinguish the singlet and triplet states, we compare our results with a weighted average of their results. For charmonium at $6/g^2 = 5.7$, they obtain $\mathcal{G}_{1L} = 0.133(4)$, and for bottomonium at $6/g^2 = 6.0$, they obtain $\mathcal{G}_{1L} = 0.144(4)$, in good agreement with the results of Table I. For these matrix elements, as with masses, most of the effect of contributions of higher order in $v^2$ is to split the results for the singlet and triplet states without shifting the weighted average.

There are some additional sources of error that we have not included in Table III. One of these is the uncertainty in the physical value of $a^{-1}$. Using the results of the NRQCD Collaboration, we find that the uncertainties in the values of the matrix elements from this source are 7% for $\mathcal{G}_1$ and 13% for $\mathcal{H}_1$ in charmonium and 13% for $\mathcal{G}_1$ and 23% for $\mathcal{H}_1$ in bottomonium. As we have already mentioned, extraction of the phenomenological matrix elements requires knowledge of the heavy-quark mass. The NRQCD Collaboration quotes an error of 4% for the $b$-quark mass, which introduces an 8% error in $\mathcal{G}_1$ and a 16% error in $\mathcal{H}_1$. For the $c$-quark mass we have no good error estimate. Further sources of uncertainty are the QCD radiative corrections to parton-level decay rates. Estimates of these uncertainties have been included in the phenomenological values of $\mathcal{H}_1$ and $\mathcal{H}_8$ for charmonium that are reported in Ref. [9]. Finally, there are the errors that arise from using quenched (rather than full) QCD, for which we have no estimates.

Let us now discuss our results. For charmonium $\mathcal{G}_1$, $\mathcal{H}_1$, and $\mathcal{H}_8/\mathcal{H}_1$ are in agreement with experiment, although both the lattice and experimental results have sizable errors. We note that we would not have found this agreement had we failed to include the perturbative corrections that relate the lattice matrix elements to the continuum ones. The quantity $\mathcal{F}_1/\mathcal{G}_1$ is poorly determined, for both charmonium and bottomonium, owing to the mixing of $\mathcal{F}_1$ with $\mathcal{G}_1$ in (7). Because $\mathcal{F}_1/(M^2\mathcal{G}_1)$ is of order $v^2 \ll 1$, a coefficient $\phi/M^2$ of order $\alpha_s$ yields a large mixing, and any uncertainties in $\phi/M^2$ are amplified in $\mathcal{F}_1/(M^2\mathcal{G}_1)$. We do learn, though, that $\mathcal{F}_1/(M^2\mathcal{G}_1)$ is no larger than $\mathcal{O}(v^2)$, in agreement with the NRQCD scaling rules [1,3]. For charmonium, $\mathcal{F}_1/\mathcal{G}_1$ is probably positive, while for bottomonium, a negative value is preferred. In the case of bottomonium, the lattice result for $\mathcal{G}_1$ is 35—40% below the experimental value, although there is good agreement between the $6/g^2 = 5.7$ and $6/g^2 = 6.0$ predictions. At least part of this discrepancy, which was first noted by the NRQCD Collaboration, is due to the quenched approximation [5,11]. Our results for the $P$-wave matrix elements for bottomonium can be translated into predictions for bottomonium decay rates [9]. In the $P$-wave case,

TABLE III. Continuum $\overline{\text{MS}}$ decay matrix elements from our lattice calculations, compared with those extracted from experimental decay rates, where available.

| | Lattice: Lattice units | Lattice: Physical units | Experiment |
|---|---|---|---|
| Charmonium $6/g^2 = 5.7$ | | | |
| $\mathcal{G}_1$ | $0.1780(3)\,(16)^{+366}_{-259}$ | $0.3312(6)\,(30)^{+681}_{-483}$ GeV$^3$ | $0.36(3)$ GeV$^3$ |
| $\mathcal{F}_1/\mathcal{G}_1$ | 0.05–0.54 | 0.07–0.82 GeV$^2$ | 0.057 GeV$^2$ |
| $\mathcal{H}_1$ | $0.0285(2)\,(27)^{+60}_{-42}$ | $0.0802(6)\,(77)^{+167}_{-118}$ GeV$^5$ | $0.077(19)\,(28)$ GeV$^5$ |
| $\mathcal{H}_8/\mathcal{H}_1$ | $0.086(1)\,(6)^{+42}_{-32}$ | $0.057(1)\,(4)^{+27}_{-21}$ GeV$^{-2}$ | $0.095(31)\,(34)$ GeV$^{-2}$ |
| Bottomonium $6/g^2 = 5.7$ | | | |
| $\mathcal{G}_1$ | $0.8279(8)\,(59)^{+1066}_{-848}$ | $2.129(2)\,(15)^{+274}_{-218}$ GeV$^3$ | $3.55(8)$ GeV$^3$ |
| $\mathcal{F}_1/\mathcal{G}_1$ | $-3.7$–$0.2$ | $-6.9$–$0.4$ GeV$^2$ | ⋯ |
| Bottomonium $6/g^2 = 6.0$ | | | |
| $\mathcal{G}_1$ | $0.1692(6)\,(14)^{+126}_{-110}$ | $2.340(8)\,(19)^{+173}_{-151}$ GeV$^3$ | $3.55(8)$ GeV$^3$ |
| $\mathcal{F}_1/\mathcal{G}_1$ | $-0.34$–$0.28$ | $-2.0$–$1.6$ GeV$^2$ | ⋯ |
| $\mathcal{H}_1$ | $0.0205(9)\,(28)^{+23}_{-19}$ | $1.63(7)\,(23)^{+19}_{-15}$ GeV$^5$ | ⋯ |
| $\mathcal{H}_8/\mathcal{H}_1$ | $0.0151(2)\,(14)^{+33}_{-29}$ | $0.00262(3)\,(24)^{+57}_{-51}$ GeV$^{-2}$ | ⋯ |

these should lead to significant new tests of the theory as the relevant experimental data become available.

As is clear from Table III, the largest uncertainties in the lattice matrix elements (aside from those due to quenching) come from neglecting higher-order corrections to the coefficients of Table II. This suggests that a useful strategy might be to use lattice methods [12] to compute the coefficients beyond leading order. In addition, one might consider the use of alternatives to the $\overline{\text{MS}}$ regularization of the continuum matrix elements so as to avoid renormalon ambiguities in the matrix elements and short-distance coefficients [12]. Note that, to the extent that we can replace $\mathcal{H}_{1L}$ and $\mathcal{H}_1$ by their vacuum saturation approximations, $\mathcal{H}_1$ and $\mathcal{H}_8$ are free of renormalon ambiguities. Of course, these ambiguities cancel in physical quantities if one works consistently to a given order in $\alpha_s$ in both the lattice-to-continuum coefficients and the short-distance coefficients.

It is an interesting fact that, for decays of $P$-wave states in both charmonium and bottomonium, the ratio of the octet matrix element to the singlet matrix element is in reasonable agreement with a crude phenomenology. This phenomenology is based on solving the one-loop evolution equation for $\mathcal{H}_8$ (Ref. [1]) and assuming that $\mathcal{H}_8$ vanishes below a scale $M_Q v$. Since the one-loop evolution contribution to the decay matrix element $\mathcal{H}_8$ is the same as that for the corresponding production matrix element $\mathcal{H}_8'$, this simple phenomenology suggests that $\mathcal{H}_8'$ is approximately equal to $\mathcal{H}_8$. For charmonium production, $\mathcal{H}_8'$ has been extracted from CDF data [13] and can also be deduced from recent CLEO data [14]. Dividing the CDF and CLEO values of $\mathcal{H}_8'$ by the phenomenological value of $\mathcal{H}_1$ given in Table III, we obtain 0.042(19) and 0.046(28) GeV$^{-2}$, respectively, which agree, within errors, with our lattice result for $\mathcal{H}_8/\mathcal{H}_1$.

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