Counting States of Near-Extremal Black Holes

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A six-dimensional black string is considered and its Bekenstein-Hawking entropy computed. It is shown that to leading order above extremality this entropy precisely counts the number of string states with the given energy and charges. This identification implies that Hawking decay of the near-extremal black string can be analyzed in string perturbation theory and is perturbatively unitary. [S0031-9007(96)01166-0]

Classical general relativity and quantum field theory in curved spacetime together provide a beautiful thermodynamic description of black holes. As Hawking showed [1], black holes radiate thermally at a temperature $T = \kappa/2\pi$, where κ is the surface gravity. The laws of thermodynamics are obeyed if one assigns an entropy to the black hole equal to one quarter of the horizon area [1,2]. However, thermodynamics is only an approximation to a more fundamental description in terms of quantum states. There have been many efforts to describe these states for black holes [3–13]. This is a difficult task since a full description requires a quantum theory of gravity.

Recently, there has been further progress in this direction. This was made possible using a new description of solitonic states in string theory [14–21]. For a particular five-dimensional extremal black hole, one can now explicitly count the number of corresponding Bogomol'nyi-Prassad-Sommerfield–saturated states in the theory with given charges and show that, for large charge, the number grows as $e^{A/4}$, where A is the horizon area [22].

In this paper we will extend this result to slightly excited, nonextremal black holes. We will show that to first order away from extremality the number of states can still be counted microscopically and continues to be given by the black hole entropy formula. The identification of extremal black hole excitations with string states enables one to use string perturbation theory to study the Hawking decay of near-extremal black holes. In particular, as we will briefly discuss in the last section, this implies that Hawking emission is a unitary process in string perturbation theory.

The five-dimensional black hole of [22] is a six-dimensional black string which winds around a compact internal circle. It was shown in [22] that the black hole states can be simply described in terms of a number of degrees of freedom living on the circle. The extremal black hole has two types of charges. One charge determines the number of degrees of freedom, and the other determines their right-moving momenta. The left-moving momenta is zero at extremality. One might expect that nonextremal black holes should correspond to keeping the same degrees of freedom, but now giving

them left-moving momenta as well. This is exactly what we find.

For our purposes it is clearer to use the six-dimensional black string description rather than the five-dimensional black hole. Below the required black string solution is discussed. The extremal solution with zero momentum has zero horizon area, indicating a nondegenerate ground state. If one adds right-moving momenta, the black string solution stays extremal, but the horizon area grows with the momenta. Dimensional reduction of this black string along its length reproduces (a slight generalization of) the five-dimensional extremal black hole in [22]. If one adds both left- and right-moving momenta, the black string becomes nonextremal, and it reduces to a nonextremal black hole. We show that the number of string states agrees precisely with that given by the black string entropy. We conclude with a brief discussion of the implications of this result in section 4.

A general black string solution.—Type IIB string theory in six dimensions contains the terms

$$\frac{1}{16\pi} \int d^6x \sqrt{-g}\left(R - (\nabla\phi)^2 - \frac{1}{12} e^{2\phi}H^2\right) \quad (1)$$

in the six-dimensional Einstein frame. H denotes the RR three-form field strength. We adopt conventions in which $G_N = 1$. We wish to consider black string solutions to (1), for which the line element can be written in the form

$$ds_6^2 = e^{2D}(dx_5 + A_\mu dx^\mu)^2 + ds_5^2, \quad (2)$$

where $\mu, \nu = 0, 1, \ldots, 4$. D and A_μ depend only on x^μ, and D tends to zero far from the string. Nonzero A_μ is required when the string carries longitudinal momentum. It is convenient to periodically identify $x_5 \sim x_5 + L$, so that the string winds along a compact dimension of asymptotic length L, which we take to be very large or infinite. The equations of motion following from (1) are equivalent to those of the five-dimensional action

$$\frac{L}{16\pi} \int d^5x \sqrt{-g}\, e^D\left(R - (\nabla\phi)^2 - \frac{2}{3}(\nabla D)^2\right.$$
$$- \frac{e^{-2D+2\phi}}{4}H_+^2 - \frac{e^{-2D-2\phi}}{4}H_-^2$$
$$\left.- \frac{e^{2D}}{4}G^2\right). \quad (3)$$

This action contains three U(1) gauge fields: $G = dA$ is the usual Kaluza-Klein field strength, H_+ derives from the reduction $H = H_+ \wedge dx^5$ [i.e., $(H_+)_{\mu\nu} = H_{\mu\nu5}$] and $H_- = e^{2\phi+D} * H$ where $*$ denotes the *five-dimensional* dual.

The six-dimensional string can carry electric charge with respect to both H_+ and H_-,

$$Q_+ \equiv \frac{1}{8} \int_{S^3} e^{-D+2\phi} * H_+ ,$$

$$Q_- \equiv \frac{1}{4\pi^2} \int_{S^3} e^{-D-2\phi} * H_- , \qquad (4)$$

It may also carry total Arnowitt-Deser-Misner (ADM) momentum P which appears in five dimensions as the charge associated with G:

$$P \equiv \frac{2\pi n}{L} = \frac{L}{16\pi} \int_{S^3} e^{3D} * G . \qquad (5)$$

We have chosen our conventions so that n and $Q_-Q_+ \equiv \frac{1}{2} Q^2$ are integers. (In the notation of [22], $n = Q_H$

and $Q^2 = Q_F^2$. The field normalization used here differs from [22].) For finite momentum density and large L, $n \gg 1$.

Black string solutions are characterized by Q_-, Q_+, n, as well as their energy density and the asymptotic value of ϕ. We are primarily interested in the black string entropy which cannot depend on the asymptotic value of the ϕ [12,23,24]. For a special asymptotic value ϕ_h, the source for ϕ (namely, H_-^2 and H_+^2) cancel exactly, and the equations of motion imply ϕ is constant everywhere. This special value is

$$e^{2\phi_h} = \frac{2Q_+}{\pi^2 Q_-} . \qquad (6)$$

In order to compute the entropy it is sufficient to consider the solutions with $\phi = \phi_h$. These are obtained by boosting the nonextremal, zero-momentum, six-dimensional black string solution found in [25]. The result is

$$\phi = \phi_h , \qquad e^{2\phi-D} * H_+ = \frac{4Q_+}{\pi^2} \epsilon_3 , \qquad e^{-2\phi-D} * H_- = 2Q_- \epsilon_3 ,$$

$$ds^2 = -\left[1 - \left(\frac{r_+^2 \cosh^2 \alpha - r_-^2 \sinh^2 \alpha}{r^2} \right) \right] dt^2 + \sinh 2\alpha \frac{r_+^2 - r_-^2}{r^2} dt dx_5 \qquad (7)$$

$$+ \left[1 - \left(\frac{r_-^2 \cosh^2 \alpha - r_+^2 \sinh^2 \alpha}{r^2} \right) \right] dx_5^2 + \left(1 - \frac{r_-^2}{r^2} \right)^{-1} \left(1 - \frac{r_+^2}{r^2} \right)^{-1} dr^2 + r^2 d\Omega_3^2 ,$$

where ϵ_3 is the volume form on the unit three-sphere and α is the boost parameter. The parameters $r_\pm$ denote the event horizon and the inner horizon, and are related to the charge by $Q^2 \equiv 2Q_+Q_- = (\pi r_+r_-)^2$. The fields D and A_μ can be read off by comparing the metric to (2). The total ADM momentum can be computed and expressed in terms of the integer n (5) with the result

$$n = \frac{L^2}{16} \sinh 2\alpha (r_+^2 - r_-^2) . \qquad (8)$$

The ADM energy of these solutions is

$$E = \frac{L\pi}{8} [2(r_+^2 + r_-^2)$$
$$+ (\cosh^2 \alpha + \sinh^2 \alpha)(r_+^2 - r_-^2)] . \qquad (9)$$

The Hawking temperature is

$$T_H = \frac{\sqrt{r_+^2 - r_-^2}}{2\pi r_+^2 \cosh \alpha} . \qquad (10)$$

The associated entropy is

$$S = \frac{A}{4} = \frac{1}{2} L\pi^2 r_+^2 \cosh \alpha \sqrt{r_+^2 - r_-^2} . \qquad (11)$$

Extremal solutions can carry all three charges Q_-, Q_+, and P, but have $T_H = 0$ and a double horizon with $r_+ =$

r_-. Such solutions are obtained from the general family of solutions (7) by taking the limit $r_+ \to r_-$ with P held fixed, which requires $\alpha \to \infty$. The resulting solutions have energy,

$$E_{\text{ext}} = \frac{LQ}{2} + \frac{2\pi n}{L} , \qquad (12)$$

and entropy [22],

$$S = \pi Q \sqrt{2n} . \qquad (13)$$

It is important to note that when $n = P = 0$ the horizon still has a small curvature of order $1/Q$ [25]. Hence α' corrections will remain negligible. The area of the horizon vanishes because of longitudinal contraction along the string.

We wish to consider solutions which correspond to low-lying, low-temperature excitations of the zero-momentum black string ground state. These are obtained by keeping α finite and expanding

$$r_\pm = r_0 \pm \epsilon , \qquad \epsilon \ll 1 , \qquad (14)$$

where $r_0 = \sqrt{Q/\pi}$. The ADM excitation energy δE above the ground-state energy $(LQ/2)$ depends on ϵ to leading order as

$$\delta E = \frac{L\pi r_0 \epsilon}{2} (\cosh^2 \alpha + \sinh^2 \alpha) . \qquad (15)$$

The entropy is given by

$$S = L\pi^2 r_0^2 \cosh \alpha \sqrt{r_0 \epsilon}. \qquad (16)$$

This can be rewritten as

$$S = \pi Q(\sqrt{2n_L} + \sqrt{2n_R}), \qquad (17)$$

where the left- and right-moving momenta of the string obey $n_R - n_L = n$ and are defined by

$$n_R \equiv \frac{L}{4\pi}(\delta E + P),$$
$$n_L \equiv \frac{L}{4\pi}(\delta E - P). \qquad (18)$$

(17) is a good approximation if both the energy density $\delta E/L$ and momentum density P/L are small. n_L and n_R can still be large if L is large. (17) incorporates the leading nontrivial behavior near extremality and describes the low-temperature black string thermodynamics. In the next section, we will show that this entropy formula is in precise accord with the statistical entropy obtained from string theory.

Counting black string microstates.—We wish to count states in string theory with the same mass and charges as the black string in the previous section. Since the black string carries Ramond-Ramond (RR) charges Q_+, Q_-, we will use the perturbative description of these states in terms of D-branes. We consider type IIB string theory compactified on T^4 or $K3$. (The following argument is independent of which space is used in the compactification.) A single extended RR string, or D-onebrane, in six dimensions carries the charge $Q_+ = 1$. The dual charge Q_- is one for a single RR fivebrane which wraps the internal four dimensions. (If the internal space is $K3$, there is an anomalous shift in the Q_- charge [19] which can be ignored for large Q.) The extremal black string solution of the previous section corresponds to a bound state of Q_+ RR strings and Q_- RR fivebranes. Since the four-dimensional compact space is assumed small, this bound state is a string in six dimensions. The entropy of such configurations may be counted as follows [22] for $Q_- = 1$. (The entropy should depend only on the product $Q_+ Q_-$. Other values of Q_-, Q_+ can be obtained by T duality, but the counting problem is different.) The strings and fivebranes do not separate in the noncompact six-dimensional spacetime, but the Q_+ RR strings are free to wander around in the internal four-dimensional space. This yields $4Q_+$ massless bosons, together with their superpartners, in the $1 + 1$ effective field theory on the string. (These correspond to fundamental open strings whose Dirichlet boundary conditions confine them to the string.) Extremal BPS configurations with nonzero momentum can be obtained by exciting only the right-moving components of these massless fields. For $n_R \gg 1$, the number of such states is given by the standard two-

dimensional entropy-energy relation $S = 2\pi\sqrt{cn_R/6}$. For $2Q^2$ species of fermions and bosons, $c = 3Q^2$ and thus

$$S = \pi Q\sqrt{2n_R}, \qquad (19)$$

in perfect agreement with (17) [22]. This result is valid in the thermodynamic limit of large n, which can always be attained for any fixed momentum density by taking L to be large.

The nonextremal case is a simple extension of this. Now we must drop the restriction to pure right movers and count states with a given (n_L, n_R). At low energies and densities the interactions between left and right movers can be ignored, and the statistical entropy is just the sum

$$S = \pi Q(\sqrt{2n_L} + \sqrt{2n_R}), \qquad (20)$$

again in perfect agreement with (17). Since the energy of the black string is proportional to L, we can get arbitrarily near extremality and remain in the thermodynamic limit by taking L sufficiently large, hence avoiding the limitations on the statistical description of near-extremal black holes discussed in [26].

Our results bear on the issue of unitary in black hole evaporation. One can view the process of scattering by an extremal black hole in terms of an absorption of the incident quanta (which excites the black hole just above extremality) followed by Hawking decay back to extremality. Since Hawking radiation is involved, it has been argued that information about the incident quanta is lost in the black hole, and unitarity is violated. However, we may alternatively describe this process in terms of string scattering by D-branes. This has been understood in some detail recently [14,16,27] and is certainly unitarity. Hence perturbative string theory provides a unitary description of scattering off certain extremal black holes.

However, this does not resolve the issue of information loss for the following reason [22,28]. The ratio of the string length to the Planck length grows as an inverse power of the string coupling. The size of an extremal RR black hole (as measured by its Schwarzschild radius) is a power of its charge times the Planck length. Hence, in string perturbation theory, strings are treated as much larger than the RR black holes. The perturbative stringy description of a RR black hole is as a D-brane with no analog of an event horizon. Proponents of unitarity violation might argue that it is not surprising that a description which cannot see the event horizon also cannot see information loss. As an analogy, perturbative unitarity of flat-space graviton scattering in string theory seems to be universally accepted, yet Hawking has argued that nonperturbatively black holes can be formed and unitarity will be violated. Further exploration of these issues is certainly in order.

The D-brane description is generally valid only for very weak coupling, $g_s < 1/Q^2$, because open string

loops couple proportionally to the number of D-branes. At stronger coupling the Schwarzschild radius becomes larger than the string scale. In this regime, the D-brane description is unreliable and the black hole description is valid. Given that the two descriptions do not appear to have an overlapping region of validity, one may wonder why our two calculations, which utilize different descriptions, are in agreement. In the extremal case discussed in [22], the topological stability of BPS states was used to argue that the results of the D-brane calculation could be extrapolated from weak to strong coupling. That argument is not directly applicable here because the states under consideration are not all BPS saturated. However, similar reasoning can be applied. The entropy computed in the D-brane picture is independent of the coupling to first order above extremality. That is because, for very large L, we are considering only long wavelength modes with very small energy densities and correspondingly small interactions. As the coupling is turned up, interactions between left- and right-moving modes of the D-brane become stronger. Nevertheless, since the leading-order low-energy result (17) is coupling independent, one expects the answer to change only if there is a phase transition which changes the number of degrees of freedom. There is no reason to expect such a transition to occur, and the remarkable agreement between the weak-coupling D-brane result and the strong-coupling black hole result is evidence that it does not occur. Going beyond the calculation described here—for example, to numerically compare the decay rate or S matrices computed in the black hole and D-brane pictures—may well require grappling with the problem of strong coupling. Perhaps string duality will be useful in this regard.

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