Thermal Ratchets in 1 + 1 Dimensions

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The notion of thermal ratchet is extended to the 1 + 1 dimensional case of an overdamped soliton-bearing theory coupled to a Gaussian source of spatiotemporal noise with finite correlation time. A stationary noise-induced current of kinks and antikinks in opposite directions is computed as a function of the noise *correlation time* and the kink (antikink) *asymmetry*. [S0031-9007(96)01228-8]

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In recent years, a number of authors [1–10] addressed the long-standing problem of how to extract useful work from a fluctuating environment [11]. While the second law of thermodynamics excludes that heat may be transformed back to mechanical work at thermal equilibrium (i.e., in the absence of temperature gradients [12]), the same restriction proves ineffective in the case of nonequilibrium thermal fluctuations: An asymmetric device (like Feynman's ratchet [12]) can rectify, indeed, symmetric *quasiequilibrium* fluctuations [2–4]. The implications of such a mechanism in transport theory are far reaching: Macroscopic currents may arise even in the absence of external forces or gradients. Consider an overdamped Brownian particle which is free to move in a large-scale homogeneous structure (i.e., periodic or random) characterized by an axial symmetry (say, its parity be broken in the x direction). The principle of detailed balance [13] teaches us that the lack of $x \to -x$ symmetry does not suffice to sustain a net average velocity in either direction. However, if stationary nonequilibrium conditions are established such that the $t \to -t$ symmetry is violated, then the onset of a net current cannot be ruled out [14]. The simplest example of noise induced transport is modeled by the stationary process

$$\dot{x} = -V'(x) + \xi(t), \qquad (1)$$

where the *asymmetric* potential $V(x)$ is periodic $V(x + 2\pi) = V(x)$, has minima at $x = (2n + 1)\pi$ with $n = 0, \pm 1, \pm 2, \ldots$, two flexural points per unit cell, and one potential barrier centered in the interval $[2n\pi, (2n + 1)\pi]$ (see potential of Fig. 1). The $t \to -t$ symmetry of the process $x(t)$ is broken by assuming that the zero mean valued Gaussian noise $\xi(t)$ is time correlated. For an Ornstein-Uhlenbeck noise

$$\langle \xi(t)\xi(0)\rangle = (D/\tau)\exp(-|t|/\tau) \qquad (2)$$

the net current $j = \langle \dot{x} \rangle$ is *positive* definite [4]. Note that such a result strongly depends both on the statistics and/or autocorrelation function of $\xi(t)$ [4,6] and on the potential function $V(x)$ [10].

Asymmetry of the metastable potential $V(x)$ and color of the noise source $\xi(t)$ are the two requisites for the stochastic ratchet (1) to function. However, a one particle model of the type (1) and (2), while serving well the purpose of illustrating the basic mechanism at work, is far too simple to be realistic [9]. In particular, there exist extended objects like dislocations in solids [15], which may be modeled as a chain of linearly coupled particles, each moving in a possibly asymmetric periodic potential. For the sake of simplicity we confine ourselves to the case of nearest neighbor coupling and assume that the chain dynamics is *overdamped*. In the continuum limit [16] we obtain what can be viewed as a 1 + 1 dimensional version of the process (1); namely, replacing x with the classical field $\phi(x,t)$ yields

$$\alpha\phi_t = c_0^2\phi_{xx} - V'[\phi] + \zeta(x,t), \qquad (3)$$

where α denotes the damping constant, c_0 is the sound speed, the function $V[\phi]$ coincides with $V(x)$ of Eq. (1), and $\zeta(x,t)$ is a Gaussian zero-mean valued noise with autocorrelation function

$$\langle \zeta(x,t)\zeta(x',t')\rangle = 2\alpha kT\,\delta(x - x')g(t - t') \qquad (4)$$

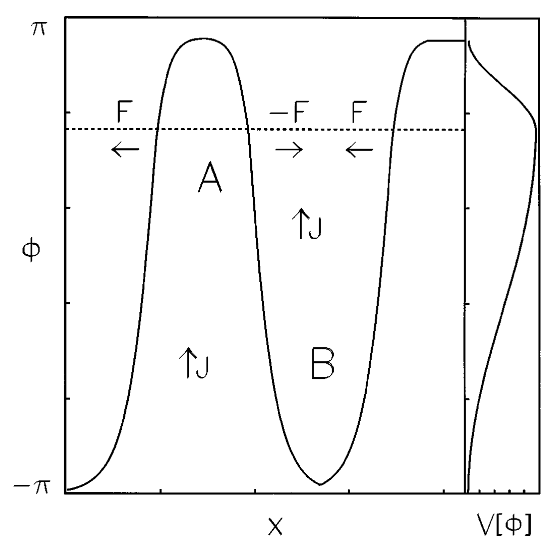

FIG. 1. Piecewise sine-Gordon potential $V[\phi]$. *A* and *B* denote nucleating pairs in the direction of increasing and decreasing ϕ, respectively. Arrows point in the direction of *F* and *J*, as labeled. The dotted line locates the potential barrier.

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and $g(t) = (1/2\tau)\exp(-|t|/\tau)$; in the limit of zero correlation time $\tau \to 0$ the noise $\zeta(x,t)$ is δ correlated both in space and in time. An elastic string $\phi(x,t)$ straddled across m potential valleys [e.g., $\phi(-\infty,t) = 0$ and $\phi(+\infty,t) = \pm 2\pi m$] bears m *geometrical* kinks or antikinks, depending on the signs $\pm$. Moreover, *thermal* kink-antikink pairs may be nucleated due to fluctuations, even in the absence of an external bias [17].

In the present Letter, we prove that, due to the potential asymmetry $V[\phi] \neq V[-\phi]$ and the noise color $\tau > 0$, a net field current $J = \lim_{L\to\infty}(1/2L)\int_{-L}^{+L}\langle\dot{\phi}(x,t)\rangle dx$, may arise as either a noise-induced drift of the kinks and antikinks with opposite stationary speeds $\pm u_F$ [18] or an imbalance of the forward versus the backward nucleation process. [For a string sitting in the nth potential valley, i.e., $\phi(x,t) = 2\pi n$, let $\Gamma_\pm$ be the number of pairs nucleated per unit of time and length in the direction $2\pi n \to 2\pi(n \pm 1)$, respectively; the net nucleation rate is $\Gamma_F = \Gamma_+ - \Gamma_-$.] These two mechanisms are not independent [17]; under stationary conditions they are related by the simple identity

$$\Gamma_F = -2n_0^2 u_F\,. \tag{5}$$

The kink (antikink) density n_0 is a function of T and τ with $n_0(T,\tau = 0) = \kappa(E_0/kT)^{1/2}\exp(-E_0/kT)$ [19]; here E_0 denotes the kink (antikink) rest energy and the constant κ depends on the function $V[\phi]$. A net field current J might represent the noise-induced transport, e.g., of dislocations gliding over asymmetric Peierls valleys on their glide plane (extended zone model) or the noise-induced rotation of optically active right- or left-handed macromolecules in solution (reduced zone model). Explicit examples of the biological relevance of this class of models are discussed in Ref. [1].

In the following we calculate the kink drift velocity u_F in the $\phi \to -\phi$ asymmetric theory (3) and (4). The kink function $\Phi(x,t) = \Phi(x - X(t))$ is assumed to be known [20]: in particular, its center of mass $X(t)$ fluctuates (in neutral equilibrium) subjected to the noise $\zeta(x,t)$ [18,21]. Moreover, for our choice of $V[\phi]$, $\Phi(x,t)$ is skewed forwards, meaning that $\Phi(x,t)$ crosses the top of the potential barrier on the rhs of its center of mass $X(t)$. On adopting a self-consistent collective variable scheme [19,21], the dynamics of the kink center of mass can be separated from the remaining degrees of freedom, thus leading to the Langevin equation of a Brownian particle with mass $M_0 = E_0/c_0^2$ and coordinate $X(t)$. According to a simple energy conservation argument [18,21], the energy dissipation rate for an individual kink coupled to the noise source $\zeta(x,t)$ is $-\int \Phi_t(x,t)\zeta(x,t)dx$, whence the force $\mathcal{F}(\tau,t)$ associated with the kink coordinate

$$\mathcal{F}(\tau,t) = \int \Phi_x(x,t)\zeta(x,t)\,dx\,. \tag{6}$$

To derive Eq. (6) we made use of the identity $\Phi_t(x,t) = -\dot{X}\Phi_x(x,t)$, implied by the very definition of the kink coordinate $X(t)$. Since $\Phi(x,t)$ depends on the perturbation $\zeta(x,t)$ through the time evolution of its center of mass, $\mathcal{F}(\tau,t)$ may be expanded perturbatively as

$$\mathcal{F}(\tau,t) = \int \Phi_x^{(0)}(x,t)\zeta(x,t)dx + \sum_n (1/n!)\int [-X(t)]^n[d^n\Phi_x^{(0)}(x,t)/dx^n]dx\,, \tag{7}$$

where $X(t) = (1/\alpha M_0)\int_0^t \Phi_x^{(0)}(x',t')\zeta(x',t')dx'dt$ and $\Phi^{(0)}(x,t) = \Phi^{(0)}(x - ut)$ denotes an *unperturbed* kink translating with constant speed u [20]. When taking the stochastic average of $\mathcal{F}(\tau,t)$ over the different realizations of $\zeta(x,t)$, we recognize immediately that the first nonvanishing term of $F(\tau) = \lim_{t\to\infty}\langle\mathcal{F}(\tau,t)\rangle$ is

$$F(\tau) = -(1/\alpha M_0)\int_0^\infty dt' \int \Phi_{xx}^{(0)}(x,t)\Phi_x^{(0)}(x',t') \times \langle\zeta(x,t)\zeta(x',t')\rangle dx'dx\,. \tag{8}$$

Here the limit $t \to \infty$ is required to eliminate transient effects due to the initial condition $X(0)$. Correspondingly, the speed u of the unperturbed $\Phi^{(0)}(x,t)$ in Eq. (8) is determined by the stationarity conditions of a dilute gas of kinks and antikinks, namely, $u = (k\overline{T}/M_0)^{1/2}$ with $\overline{T} = \overline{T}(\tau)$ [22]. In view of the $\zeta(x,t)$ autocorrelation function (4), Eq. (8) for the noise-induced force acting on a kink simplifies to

$$F(\tau) = -2(kT/M_0)\int_0^\infty g(\Delta)f(\Delta)\,d\Delta\,, \tag{9}$$

where $g(\Delta) = (1/2\tau)\exp(-|\Delta|/\tau)$ and

$$f(\Delta) = \int \Phi_x^{(0)}(x,0)\Phi_{xx}^{(0)}(x,\Delta)\,dx\,, \tag{10a}$$

with [23]

$$f(\Delta) = -f(-\Delta)\,. \tag{10b}$$

The fluctuating component of $\mathcal{F}(\tau,t)$ amounts to a zero mean valued random force $\xi(t)$ acting on the kink center of mass. On restricting our analysis to the first term on the rhs of Eq. (7), we can easily check that $\xi(t)$ is a Gaussian noise with correlation time $\overline{\tau}$ given by $\overline{\tau} = \min\{\tau, d(M_0/k\overline{T})^{1/2}\}$, where d denotes an appropriate kink-size scale [22]. Hence $X(t)$ obeys the following Langevin equation:

$$M_0\ddot{X} = -\alpha M_0\dot{X} + F(\tau) + \xi(t)\,. \tag{11}$$

Equations (9)–(11) summarize the main result of the present investigation; a nontrivial prediction, indeed, provided that integral (9) is not identically zero.

Before exploring the possibility that $F(\tau) \neq 0$, we consider two trivial limits: (1) *White noise* $\tau = 0$ (or equilibrium fluctuations). In this case $g(\Delta) = \delta(\Delta)$ and $F(0) = -2(kT/M_0)f(0)$. Independently of the potential symmetry $f(0) = (1/2)\Phi_x^{(0)2}(x)|_{-\infty}^{+\infty} = 0$, whence $F(0) = 0$; no drift for a single kink and therefore, no net field current can be induced by a white noise source. (2) *Symmetric potential* $V[\phi] = V[-\phi]$. The symmetry of $V[\phi]$ implies that $\Phi_x^{(0)}(x,t) = \Phi_x^{(0)}(-x,t)$ and $\Phi_{xx}^{(0)}(x,t) = -\Phi_{xx}^{(0)}(-x,t)$, whence $f(\Delta) = 0$ and, therefore, $F(\tau) = 0$. As expected, a symmetric soliton-bearing theory cannot rectify a symmetric spatiotemporal noisy signal, no matter what its correlation time.

We address now the general case of an asymmetric potential $V[\phi]$ considered above (see also Fig. 1), the function $f(\Delta)$ of Eq. (10), far from being identically zero, admits one node at $\Delta = 0$, is *positive definite* for $\Delta > 0$, and vanishes for $\Delta \to +\infty$. Recalling that $g(\Delta)$ is symmetric, definite positive, and vanishes for $\Delta \to \pm\infty$, we conclude that the integral in Eq. (9) is definite positive, too, and therefore $F(\tau) < 0$. It follows immediately that in the soliton-bearing theory of Fig. 1, the stationary noise-induced drift speed of a single kink is negative, namely from Eq. (11),

$$u_F = F(\tau)/\alpha M_0 < 0\,. \tag{12}$$

The consequences of Eqs. (9)–(12) deserve an accurate analysis.

(i) The ensuing net ϕ current can be readily expressed in terms of u_F and n_0 [17], that is $J = -(2\pi)2n_0u_F$, whence

$$J(\tau) = -(4\pi n_0/\alpha M_0)F(\tau) > 0\,. \tag{13}$$

The noise-induced current (13) is *positive* as in the 0 + 1 dimensional models of Refs. [2,4]. A qualitative explanation of such a behavior runs as follows: Any time the potential $V[\phi]$ is tilted to the right, the position of each barrier shifts to the left and vice versa. However, due to the asymmetry of $V[\phi]$, the shifts to the left are certainly more pronounced than to the right (see Fig. 1). Noting that the kink center of mass is located to the left of the corresponding potential barrier, it is clear that $\Phi(x,t)$ undergoes larger shifts to the left than to the right (the noise being symmetric), i.e., $u_F < 0$. This argument applies only if the tilt reverses sign over a finite time scale, that is, for time correlated noise sources. The stationary drift of a kink toward the string left end point causes a global advance of the string itself toward larger ϕ values, that is $J > 0$. The same argument applies to the antikinks, which drift then to the right with the same consequences on the drift of the string.

(ii) Although the force (9) can be determined explicitly only after the shape of the skewed kink $\Phi^{(0)}(x,t)$ (i.e., the asymmetric potential $V[\phi]$) has been chosen, two important limits of the function $F(\tau)$ can be discussed analytically. In the *weak color* limit $\tau \to 0$ the function $g(\Delta)$ peaks around $\Delta = 0$ so that the integral (9) is dominated by the behavior of $f(\Delta)$ at the origin. On expanding Eq. (10a) in powers of Δ, one finds that for small Δ values $f(\Delta) \simeq \Delta \int \Phi_{xx}^{(0)2}(x)\,dx$, whence $F(\tau)$ and, consequently, $J(\tau)$ turn out to be linear in τ. It is worthwhile to remember that for the related 0 + 1 dimensional process (1)–(2), the noise-induced current in the neighborhood of $\tau = 0$ was estimated to be order τ^2 or higher [4]. In the *strong color* limit $\tau \to \infty$, we replace $g(\Delta)$ by $(1/2\tau)[1 - |\Delta|/\tau]$. Recalling that $\int_0^\infty f(\Delta)d\Delta > 0$, we recover that $F(\tau)$ is proportional to $1/\tau$, while the τ dependence of $J(\tau)$ is controlled by the exponential decay of $n_0(T,\tau)$ [22]. Furthermore, the noise-induced current $J(\tau)$ attains a maximum for a value of τ that depends crucially on the asymmetry of the theory. For particular potential shapes the current inversion phenomenon [10] is not ruled out either.

(iii) On making use of Eqs. (5) and (12) we conclude that $\Gamma_+ > \Gamma_-$: Nonequilibrium thermal nucleation in an asymmetric soliton-bearing theory is favored in the direction of the net field current J. The drawings of Fig. 1 illustrate such a mechanism. The bubble A represents a nucleating pair in the direction of increasing ϕ; it is approximated by the linear superposition of a kink on the left and an antikink on the right-hand side. The noise induced forces (9) tend to pull the nucleating partners apart, thus helping the nucleation process. Vice versa, we realize by inspection that the noise induced forces (9) oppose the nucleation process represented by the bubble B.

The reader might be tempted to exploit the traveling nature of the solitary wave $\Phi^{(0)}(x,t) = \Phi^{(0)}(x - ut)$ to extend our approach to the case of a spatially correlated noise source $\zeta(x,t)$. As a first attempt, one could try to replace the autocorrelation function (4) by

$$\langle\zeta(x,t)\zeta(x',t')\rangle = 2\alpha kT\delta(t - t')g(x - x')\,, \tag{14}$$

with $g(x) = (1/2\lambda)\exp(-|x|/\lambda)$ defining a spatial correlation length λ [18,24]. Indeed, a simple calculation yields

$$F(\lambda) = -2(kT/M_0)\int_{-\infty}^{+\infty} g(\Delta)f(\Delta)d\Delta\,, \tag{15}$$

with $f(\Delta)$ given by Eq. (10). However, contrary to Eq. (9) for $F(\tau)$, the integration over Δ is extended here to the entire interval $(-\infty, +\infty)$; in view of the symmetry relation $f(\Delta) = -f(\Delta)$, one concludes that $F(\lambda) = 0$ for any choice of $V[\phi]$.

In conclusion, we proved that the notion of thermal ratchet [2,4] can be extended to the 1 + 1 dimensional case of asymmetric soliton-bearing theories. Such nonequilibrium noise-induced currents could provide an alternative rectification mechanism for the modeling of solitonic fluxes in such magnetic devices as long Josephson junction transmission lines [21] and magnetically ordered crystals [25].

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[23] On integrating by parts the definition (10a) of $f(-\Delta)$, we obtain $f(-\Delta) = -\int \Phi_{xx}^{(0)}(x,0)\Phi_x^{(0)}(x,-\Delta)\, dx$. [Note that $\Phi_x^{(0)}(\pm\infty, t) = 0$.] On shifting the x origin $x \to x - u\Delta$, we translate the solitary wave $\Phi^{(0)}(x - ut)$ by $u\Delta$, whence $f(\Delta) = -f(-\Delta)$.
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