Complementarity and Fundamental Limit in Precision Phase Measurement

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By analyzing a single-photon interference experiment together with a quantum nondemolition measurement scheme for the which-path information, we prove that given the total mean number $\langle n\rangle$ of available photons, the fundamental limit in precision measurement of a phase shift is the Heisenberg limit, i.e., $1/\langle n\rangle$. The analysis is based on the complementarity principle and is independent of the scheme for the measurement of the phase shift. We also show that to achieve the Heisenberg limit states with photon number fluctuations of the order of or larger than the mean photon number have to be exploited. [S0031-9007(96)01164-7]

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The ability to resolve an extremely small phase shift is not only of great significance in technological advances but also poses a strong challenge for modern quantum measurement theory. With an unlimited resource of energy, it is possible to measure a phase shift to arbitrary precision. In practice, however, the total amount of available energy is always finite. So any analysis of the sensitivity of phase measurement has to be made under such a constraint. The traditional argument for the limit in a precision phase measurement stems from the Heisenberg uncertainty relation for the fluctuations of phase ϕ and photon number n [1],

$$\Delta\phi\Delta n \geq 1\,, \tag{1}$$

where $\Delta n \equiv \sqrt{\langle\Delta^2 n\rangle}$ and $\Delta\phi$ characterizes the phase fluctuation. Thus shot noise ($\langle\Delta^2 n\rangle \sim \langle n\rangle$) due to particle nature of light will place a limit on how precise one can measure a small phase shift,

$$\Delta\phi \geq \frac{1}{\sqrt{\langle n\rangle}}\,. \tag{2}$$

However, shot noise is not the same noise for all quantum systems. Quantum mechanics does not set any restriction on Δn. Naively, one would argue that because of the energy constraint, Δn should be bounded by the mean number of photons, that is, $\langle\Delta^2 n\rangle \sim O(\langle n\rangle^2)$. Thus given the total mean number of photons, the limit on precision phase measurement should be the so-called Heisenberg limit,

$$\Delta\phi \geq \frac{1}{\langle n\rangle}\,. \tag{3}$$

Indeed, all the best measurement schemes discovered so far have not been able to surpass this limit [2–5].

On the other hand, Shapiro *et al.* [6,7] recently proposed the following state:

$$|\Psi\rangle = A\sum_{m=0}^{M}\frac{1}{m+1}|m\rangle \ \left(M \gg 1,\ A \simeq \sqrt{6/\pi^2}\right), \tag{4}$$

which they claim can achieve $1/\langle n\rangle^2$ precision in the measurement of a phase shift. But some difficulties associated with the phase distribution of this state prevent it from achieving the promised performance [8–10]. In spite of this, it is interesting to note that for this state we have $\langle\Delta^2 n\rangle = A^2 M$ while $\langle n\rangle \simeq A^2 \ln M$ so that $\langle\Delta^2 n\rangle \sim A^2\exp(\langle n\rangle/A^2)$. It is also possible to find some other states with $\langle\Delta^2 n\rangle \sim \langle n\rangle^\gamma (\gamma > 2)$. Thus, previous argument leading to Eq. (3) for the Heisenberg limit does not hold. Furthermore, the Heisenberg uncertainty relation in Eq. (1) is derived on the basis of operator algebra in quantum mechanics [1], but it has been proved that there does not exist a Hermitian operator for phase in quantum mechanics [11]. In fact, Eq. (1) does not hold in general. For example, Eq. (1) is violated for the vacuum state. Therefore, the whole argument based on the Heisenberg uncertainty relation in Eq. (1) does not hold in general and the question remains: What is the limit on the sensitivity in a precision phase measurement, given the available total mean number of photons?

In this Letter, I will discuss the phase problem through a different approach without the need of a phase operator. I will prove by a simple argument regarding the change of state produced by a phase shift that the ultimate limit on the sensitivity of a precision phase measurement is the Heisenberg limit. This argument is further supported by an analysis of a single-photon interferometer coupled through the optical Kerr effect to another single mode probe field, on which a phase measurement is performed. The precision of the phase measurement is the main concern of the analysis. We are able to connect, by the complementarity principle, the visibility of the single-photon interferometer and the minimum detectable phase shift that can be measured with the probe field. We will derive a necessary condition for those states that can achieve the Heisenberg limit in a precision phase measurement.

First of all, since phase is a relative quantity and its measurement has to rely on the comparison with other fields, let us consider a single-mode field A, which is correlated with other fields denoted as B for phase comparison. The most general state describing fields A, B

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has the form of

$$|\Phi\rangle = \sum_{m,\lambda} C_m(\lambda)\,|m\rangle_A|\lambda\rangle_B\,, \tag{5}$$

where the $C_m(\lambda)$ are normalized coefficients in the basis of the Fock states $\{|m\rangle_A|\lambda\rangle_B\}$. Assume that the field A (characterized by the creation operator $\hat{A}^\dagger$) undergoes a phase shift δ (Fig. 1). In quantum mechanics, the phase shift can be modeled as a parameter in the free propagation of field A and is associated with a unitary propagation operator

$$\hat{U}_\delta = e^{i\delta\hat{A}^\dagger\hat{A}}.$$

Thus the state after the phase shift is simply

$$|\Phi'\rangle = \hat{U}_\delta|\Phi\rangle = \sum_{m,\lambda} C_m(\lambda)e^{i\delta m}\,|m\rangle_A|\lambda\rangle_B\,. \tag{6}$$

Notice that the phase shift introduced here is not defined through a phase operator. Equation (6) can be rearranged to become

$$|\Phi'\rangle = |\Phi\rangle + |\Delta\Phi\rangle\,, \tag{7a}$$

with

$$|\Delta\Phi\rangle = \sum_{m,\lambda} C_m(\lambda)\,(e^{im\delta} - 1)\,|m\rangle_A|\lambda\rangle_B\,. \tag{7b}$$

Our goal is to measure the phase shift δ through the detection of the change $|\Delta\Phi\rangle$ in the state. Obviously, if the change in the state is small so that we are unable to detect it, it will be impossible to detect the phase shift δ. The magnitude of the change in the state is the norm of $|\Delta\Phi\rangle$,

$$\|\,|\Delta\Phi\rangle\|^2 \equiv \langle\Delta\Phi\,|\,\Delta\Phi\rangle = 4\sum_m P_m \sin^2(m\delta/2)\,, \tag{8}$$

where $P_m = \sum_\lambda |C_m(\lambda)|^2$ is the photon number distribution for the field A. Now, let us use the inequalities $|\sin x| \le 1$ and $|\sin x| < x$ to rewrite Eq. (8) as

$$\|\,|\Delta\Phi\rangle\|^2 \le 4\sum_m P_m|\sin(m\delta/2)| < 2\langle n\rangle\delta\,, \tag{9}$$

where $\langle n\rangle$ is the total mean number of photon in field A. From Eq. (9), we find that the minimum detectable phase shift is at least $O(1/\langle n\rangle)$. For, otherwise, we will have $\langle n\rangle\delta \ll 1$ when $\langle n\rangle \gg 1$, which indicates from Eq. (9) that $\|\,|\Delta\Phi\rangle\|^2 \ll 1$ and the change in the state is undetectable.

Next, we will apply the principle of complementarity in quantum interference to further prove that the quantity $\|\,|\Delta\Phi\rangle\|$ is indeed a good measure of whether the phase shift δ is detectable and that the minimum detectable phase shift is $1/\langle n\rangle$. The complementarity principle of quantum mechanics [12], when applied to the phenomena of interference, states that if it is possible to find the which-path information for the two possible interfering paths of a particle, the interference effect will disappear. In a more quantitative description, the degree of the interference effect (e.g., the visibility of the interference pattern) will depend on the precision of our knowledge about which path the particle goes through. Although a phase measurement relies on interferometry, it is based on the existence and not on the disappearance of interference effect. It seems that the question of how precise a phase shift can be measured is not related to the complementarity principle. On the other hand, the photon number can be determined by quantum nondemolition (QND) measurement without destroying the photons. It is known [13,14] that the optical Kerr effect can be used to implement a QND measurement of the photon number. In this case, the measured photon imposes a phase shift on another beam called the probe beam. Measurement of the phase shift on the probe beam provides the information about the photon number. The knowledge of the exact photon number in one path of a single-photon interferometer determines which path the photon follows and thus destroys the interference effect. Therefore, the accuracy of the phase measurement on the probe field will influence the result of the interference.

Consider a Mach-Zehnder interferometer with a phase difference of ϕ as shown in Fig. 2. A single-photon state is sent in one of the input ports. Without any device in the paths of the interferometer, the output of the interferometer will exhibit a sinusoidal modulation as a function of ϕ with a visibility of one. When we place a device in one of the paths, say path a (characterized by the creation operator $\hat{a}$), to find out which path the photon passes, this usually will disturb the interferometer and, depending on the device, the visibility of the interference will degrade. In order to preserve the photon, we will use the Kerr effect as the QND

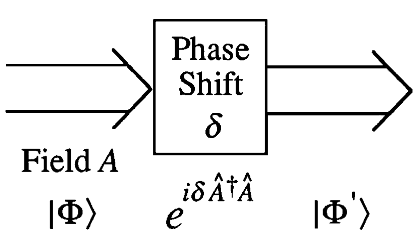

FIG. 1. General scheme for the generation of a phase shift in field A.

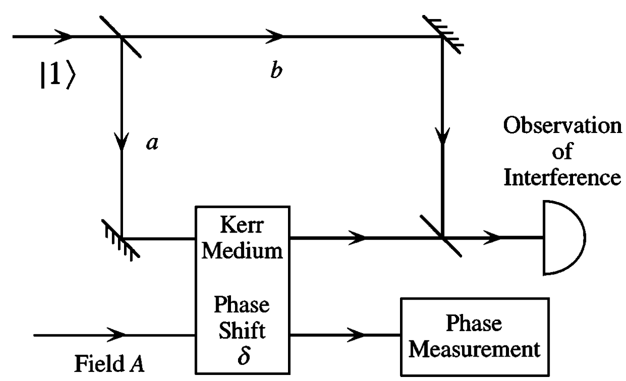

FIG. 2. Single photon interferometer with a Kerr medium for QND measurement of the photon number in the path a. Determination of the photon number in path a through accurate measurement of the phase shift in field A induced by a single photon in field a will destroy the interference effect.

scheme to determine the photon number (Fig. 2). In the QND scheme, the signal field a and the probe field A are coupled through a Kerr medium and the state evolution is determined by the unitary operator [14]

$$\hat{\mathcal{U}}_{\rm QND} = e^{i\delta\hat{a}^\dagger\hat{a}\hat{A}^\dagger\hat{A}}, \tag{10}$$

where δ is a parameter characterizing the strength of the interaction. To see further the physical meaning of δ, let the input state to the QND device be a single photon state for field a and a general state $|\Phi\rangle$ given in Eq. (5) for field A. Then the output state after the QND interaction is

$$\hat{\mathcal{U}}_{\rm QND}|1\rangle_a|\Phi\rangle_A = |1\rangle_a e^{i\delta\hat{A}^\dagger\hat{A}}|\Phi\rangle_A\,, \tag{11}$$

Thus, from Eq. (6), the quantity δ is the phase shift imposed on the probe field A due to the input of a single photon in field a. Measurement is then performed on the field A to estimate the phase shift (Fig. 2). This can be achieved by performing a homodyne detection or other type of interferometric method. But the exact detail of the phase measurement is not our concern here. The following argument applies to any scheme of phase measurement. If we can detect the phase shift of size δ in field A with whatever means, we will be able to tell whether the photon in the interferometer passes the path a or not. Hence, according to the complementarity principle, the interference effect will disappear. On the other hand, if we can observe a 100% visibility in the single-photon interferometer, it is impossible to detect the phase shift δ in field A no matter what kind of method or strategy we use for the extraction of the phase shift. Therefore, the visibility of the interferometer is directly related to the ability to resolve the phase shift δ in field A. Notice that the phase of concern is that of the probe field A in QND measurement but not ϕ of the interferometer. In fact, we are only interested in the visibility of the interferometer.

Let us now examine the visibility of the single-photon interferometer with the QND device discussed above in path a. We will find out its relation with the state in field A. Assume a single-photon state is sent to one of the input ports of the interferometer. After the first beam splitter, the state for the system becomes

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\,(|1\rangle_a|0\rangle_b + e^{i\phi}|0\rangle_a|1\rangle_b)|\Phi\rangle_A\,. \tag{12a}$$

After passing the Kerr medium, the state of the system has the form of

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\,(|1\rangle_a|0\rangle_b e^{i\delta\hat{A}^\dagger\hat{A}}|\Phi\rangle_A + e^{i\phi}|0\rangle_a|1\rangle_b)|\Phi\rangle_A)\,. \tag{12b}$$

From this state, we can calculate the probability of detecting a photon at one output port. It has the form of

$$P = \tfrac{1}{2}[1 \pm |\langle\Phi|e^{i\delta\hat{A}^\dagger\hat{A}}|\Phi\rangle|\cos(\phi - \epsilon)]\,, \tag{13}$$

where ϵ is the phase of $\langle\Phi|e^{i\delta\hat{A}^\dagger\hat{A}}|\Phi\rangle$. Therefore the visibility of the interference pattern is

$$v = \langle\Phi|e^{i\delta\hat{A}^\dagger\hat{A}}|\Phi\rangle| = |\langle\Phi\mid\Phi'\rangle|\,. \tag{14}$$

The above relation was also derived by Sanders and Milburn [15]. Although Eq. (14) is derived for the single photon input, it can easily be shown that even with an arbitrary input state to the interferometer, the visibility still takes the form of Eq. (14). From Eq. (5), we can find the explicit form of the visibility. For the purpose of comparison with the unit visibility, let us calculate the quantity $1 - v$ as follows:

$$\begin{aligned} 1 - v = 1 - |1 + \langle\Phi\mid\Delta\Phi\rangle| &\le |\langle\Phi\mid\Delta\Phi\rangle| \\ &\le \|\,|\Delta\Phi\rangle\|\,, \end{aligned} \tag{15a}$$

where Schwartz inequality is used, or more explicitly,

$$\begin{aligned} 1 - v \le |\langle\Phi\mid\Delta\Phi\rangle| &= 2\left|\sum_m P_m e^{im\delta/2}\sin m\delta/2\right| \\ &\le 2\sum_m P_m|\sin m\delta/2|\,. \end{aligned} \tag{15b}$$

By using the inequality $\sin x < x$ in expression (15b), we end up with the following inequality:

$$1 - v < \langle n\rangle\delta \quad \text{or} \quad \delta > (1 - v)/\langle n\rangle\,. \tag{16}$$

which sets a lower limit on the minimum detectable phase shift, given the total mean number of photon available in the field A. When it is possible to resolve the phase shift with the field A, then we can tell whether the photon entering the interferometer passes through path a or not. Since we know the which-path information, according to complementarity principle, the interference effect in the interferometer will disappear or equivalently, $v \sim 0$. Thus from Eq. (16), we find that the minimum detectable phase shift in field A is of the order of $1/\langle n\rangle$ or the Heisenberg limit.

Before we go any further, let us consider some examples. The first one is the coherent state $|\alpha\rangle$. It is easy to find from Eq. (14) that $v = |\exp[(e^{i\delta} - 1)|\alpha|^2]| \simeq \exp(-\langle n\rangle\delta^2/2)$. If the phase shift δ can be detected, visibility v must be significantly different from one, thus $\delta_m \sim 1/\sqrt{\langle n\rangle}$, which is the well-known limit for coherent state interferometry [2]. For the two-photon interferometry involving a single-mode squeezed state or a two-mode squeezed state, we find

$$v_s = \frac{1}{[(2\langle n\rangle + 1)^2\sin^2\delta + \cos^2\delta]^{1/4}} \tag{17a}$$

for the former and

$$v_t = \frac{1}{[(\langle n\rangle + 1 - \langle n\rangle\cos\delta)^2 + \langle n\rangle^2\sin^2\delta]^{1/2}} \tag{17b}$$

for the latter. Both v_s and v_t are significantly different from 1 only if $\delta > 1/\langle n\rangle$. Thus both states can be used to achieve the Heisenberg limit in a precision measurement [2,3].

As another example, let us consider the state given in Eq. (4). It can be shown that $v \approx 1 - 6\delta/\pi$, when

$\langle n\rangle \to \infty$, $\delta \ll 1$. Therefore, for $\delta \ll 1$, $v \approx 1$, and it is impossible to detect a small phase shift even if we have an infinite amount of energy. Hence, the state in Eq. (4) is not a good state for precision phase measurement.

What kind of state can achieve the Heisenberg limit? Intuitively, we find from the Heisenberg uncertainty relation for phase and photon number in Eq. (1) that the state must have large photon number fluctuations with $\langle \Delta^2 n\rangle \geq \langle n\rangle^2$. However, since Eq. (1) is not valid in general, this argument does not hold. In the following, I will use the same argument that leads to Eq. (16) to prove that the intuitive argument above is actually correct; that is, the Heisenberg limit cannot be achieved for those states with $\langle \Delta^2 n\rangle \ll \langle n\rangle^2$ for large $\langle n\rangle$.

Consider Eqs. (14) and (15) for the visibility of the interferometer,

$$v = \left| \sum_m P_m e^{im\delta} \right| . \tag{18}$$

Assume field A experiences a phase shift of $\delta \sim 1/\langle n\rangle$ due to the passing of a single photon in path a and the field A is in some state with $\langle \Delta^2 n\rangle \ll \langle n\rangle^2$. Since P_m is a distribution with a width of $\sqrt{\langle \Delta^2 n\rangle} \ll \langle n\rangle$, the photon probability distribution P_m is significantly different from zero only for those m with $|m - \langle n\rangle| \lesssim \sqrt{\langle \Delta^2 n\rangle}$. Therefore, most of the contributions to the summation in Eq. (18) come from terms with $|m - \langle n\rangle| \lesssim \sqrt{\langle \Delta^2 n\rangle}$ and Eq. (18) becomes

$$v \approx \left| \sum_{|m-\langle n\rangle| \lesssim \sqrt{\langle \Delta^2 n\rangle}} P_m e^{im\delta} \right| . \tag{19}$$

On the other hand, because $\sqrt{\langle \Delta^2 n\rangle} \ll \langle n\rangle$ and $\delta \sim 1/\langle n\rangle$, we have $\delta\sqrt{\langle \Delta^2 n\rangle} \ll 1$ and therefore we can approximate $e^{im\delta}$ in Eq. (19) with $e^{i\langle n\rangle\delta}$. Equation (19) then becomes

$$v \approx \left| \sum_m P_m e^{i\langle n\rangle\delta} \right| = 1 . \tag{20}$$

Hence, the interference pattern has a visibility of one and from the complementarity principle, we cannot tell which path the photon passes in the interferometer, which means that we cannot resolve a phase shift of $\delta \sim 1/\langle n\rangle$ from the measurement on field A in this kind of state. So, those states with $\sqrt{\langle \Delta^2 n\rangle} \ll \langle n\rangle$ at large $\langle n\rangle$ cannot achieve the Heisenberg limit in a precision phase measurement. Thus the necessary condition is $\sqrt{\langle \Delta^2 n\rangle} \gtrsim \langle n\rangle$ for the states to achieve the Heisenberg limit.

In summary, we have proved with a simple argument that the sensitivity in precision phase measurement is limited by $1/\langle n\rangle$ given the total mean photon number of $\langle n\rangle$. The argument is based on the general principle of complementarity in quantum interference and thus applies to any scheme of phase measurement. We have also found a necessary condition for the state that can achieve the fundamental limit.

In applying the complementarity principle, we find that the existence of field A, which has the ability to resolve the phase shift, is enough to destroy the interference pattern in the single-photon interferometer. No actual measurement of the phase shift is required to be performed. Thus mere possibility of distinguishing the two interfering paths is enough to make interference effect disappear. This phenomenon can also be understood from the backaction by field A on the phase of field a without the help of the complementarity principle. Because of backaction, the photon number fluctuation of field A imposes a random phase shift on field a in exactly the same way whereby field a does to field A. Such random phase shift will wash out the interference pattern as long as field A passes through the QND device. That is another reason why states with a large photon number fluctuation are needed in field A for the visibility to disappear.

Although the above conclusions are drawn from the derivation on the pure state $|\Phi\rangle$ for field A, it is straightforward to show that Eq. (18) stands even for mixed state [15]. Therefore, all the conclusions above apply equally well to mixed states.

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