PHYSICAL REVIEW LETTERS

VOLUME 77 | 16 SEPTEMBER 1996 | NUMBER 12

Asymptotic Integrability of Water Waves

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(Received 2 October 1995)

The asymptotic integrability of the idealized water waves is formally established. Namely, it is shown that in the small amplitude, long wave limit there exists an explicit transformation which maps these equations to a system of two integrable equations. It is also shown that the concepts of master symmetries and of bi-Hamiltonian structures can be used to obtain similar results for other physical systems. [S0031-9007(96)01167-2]

PACS numbers: 03.40.Gc, 47.35.+i, ,

One of the most extensively studied physical systems is the motion of a two-dimensional, inviscid, incompressible fluid. Let us consider the simplest possible case of a body of water lying above a horizontal flat bottom located at $y = -h_0$, h_0 constant, and let there be air above the water. For such a system if the vorticity is zero initially, it remains zero. The irrotational flow is characterized by two parameters, $A = a/h_0$ and $B = h_0^2/l^2$, where a and l are typical values of the amplitude and of the wavelength of the waves. Let the dimensionless quantities $\eta(x,t)$ and $\varphi(x,t)$ denote the position of the free surface and the velocity potential, respectively. The function $\varphi(x,t)$ can be expanded in the form $\varphi = \sum_0^\infty (-B)^m (1 + A\eta)^{2m} f_{2m}/(2m)!$, where $f_m = \partial^m f/\partial x^m$. We assume that $O(B) = O(A)$, and then, without loss of generality, we let $A = 2\epsilon/3$ and $B = 6\epsilon$. The functions $\eta(x,t)$ and $\omega(x,t) = f_x$ satisfy [1]

$$\eta_t + \omega_x + \tfrac{2}{3}\epsilon(\eta\omega)_x - \epsilon\omega_{xxx} + O(\epsilon^2) = 0, \quad (1a)$$

$$\omega_t + \eta_x + \tfrac{2}{3}\epsilon\omega\omega_x - 3\epsilon\omega_{xxt} + O(\epsilon^2) = 0. \quad (1b)$$

Let $K(v)$ denote

$$K(v) = v_{xxx} + 6vv_x. \quad (2)$$

Under the additional assumption that waves travel only in one direction, the water wave equations reduce to [1]

$$\eta_t + \eta_x + \epsilon K(\eta) + \epsilon^2(\alpha_1\eta_{xxxxx} + \alpha_2\eta\eta_{xxx} + \alpha_3\eta_x\eta_{xx} + \alpha_4\eta^2\eta_x) + O(\epsilon^3) = 0, \quad (3)$$

where $\alpha_1 = 19/10$, $\alpha_2 = 10$, $\alpha_3 = 23$, $\alpha_4 = -6$. Using a moving frame of reference and rescaling x and t, Eq. (3) reduces to

$$\eta_t + K(\eta) + \epsilon(\alpha_1\eta_{xxxxx} + \alpha_2\eta\eta_{xxx} + \alpha_3\eta_x\eta_{xx} + \alpha_4\eta^2\eta_x) + O(\epsilon^2) = 0. \quad (4)$$

As $\epsilon \to 0$, Eq. (4) becomes the Korteweg–de Vries (KdV) equation, which is an integrable equation [2]. Thus, unidirectional idealized water waves are *asymptotically integrable to $O(\epsilon)$* [3]. Kodama [4] has formally extended the asymptotic integrability of this system to $O(\epsilon^2)$: He has found an explicit transformation which maps Eq. (4) to the integrable equation $v_t + K(v) + \epsilon\alpha_1 K_1(v) = 0$, where $K_1(v)$ is the next commuting flow of the KdV hierarchy, i.e.,

$$K_1(v) = v_{xxxxx} + 10vv_{xxx} + 20v_xv_{xx} + 30v^2v_x. \quad (5)$$

In this Letter we present the following: (i) Present a generalization of Kodama's transformation which maps

Eq. (4) to the KdV equation itself. Furthermore, we show that Eq. (4) can be mapped to several other integrable equations. These equations are integrable generalizations of the KdV equation and of the Gardner equation (a linear combination of KdV and of the modified KdV equations). (ii) Establish formally the asymptotic integrability of two-dimensional water waves to $O(\epsilon^2)$: We use certain bi-Hamiltonian structures to derive a new system of two integrable equations, and then present an explicit transformation which maps Eqs. (1) to this new integrable system. (iii) Discuss how the concepts of master symmetries and of bi-Hamiltonian structures can be used to obtain similar results for other physical systems.

Proposition 1.—(i) Let v solve the KdV equation $v_t + K(v) = 0$, where $K(v)$ is defined in Eq. (2). Let η be defined by

$$\eta = v + \epsilon[\lambda_1 v^2 + \lambda_2 v_{xx} + \lambda_3 v_x \partial^{-1} v + \lambda_4 x K(v)], \tag{6}$$

where $\lambda_1 = 59/15$, $\lambda_2 = 22/15$, $\lambda_3 = 26/15$, $\lambda_4 = -19/30$, and ∂^{-1} denotes integration with respect to x. Then η solves Eq. (4). This result is equivalent to the following: Let $v(x,T)$ solve the KdV equation, where $T = t - \lambda_4 \epsilon x$, $\lambda_4 = -19/30$. Let η be defined by

$$\eta = v + \epsilon(\lambda_1 v^2 + \lambda_2 v_{xx} + \lambda_3 v_x \partial^{-1} v), \tag{7}$$

where $\lambda_1 = 59/15$, $\lambda_2 = 22/15$, $\lambda_3 = 26/15$. Then η solves Eq. (4).

(ii) Let v solve the generalized integrable KdV equation

$$v_t + v_{xxx} + 6 v v_x + \nu\epsilon(v_{xxt} + 2 v v_{xxx} + 4 v_x v_{xx}) = 0. \tag{8}$$

Let η be defined by Eq. (7), where $\lambda_1 = 7/5$, $\lambda_2 = 1/5$, $\lambda_3 = -4/5$, and let $\nu = -19/10$. Then η solves Eq. (4).

(iii) Let v solve the generalized Gardner equation, i.e., the equation obtained by adding the terms

$$3\rho\epsilon v^2 v_x + \rho\nu\epsilon^2(v^2 v_{xxx} + 4 v v_x v_{xx} + v_x^3) + \rho\nu^2\epsilon^3(v_x^2 v_{xxx} + 2 v_x v_{xx}^2),$$

to the left-hand side of Eq. (8). Let η be defined by $\eta = v + \epsilon(\lambda_1 v^2 + \lambda_2 v_x \partial^{-1} v)$, where $\lambda_1 = 26/3$, $\lambda_2 = -4/5$, and let $\nu = -19/10$, $\rho = 218/15$. Then η solves Eq. (4).

Proposition 2.—Let $\epsilon, \beta_1, \beta_2, \beta_3, \beta_4$ be arbitrary constants, and let $\beta_1^2 \neq 1$. The system of the equations

$$u_t + v_x + \epsilon[(3\beta_1 + 2\beta_4)\beta_3 u u_x + (2 + \beta_1\beta_4)\beta_3(uv)_x + \beta_1\beta_3 v v_x + (\beta_1 + \beta_4)\beta_2 u_{xxx} + (1 + \beta_1\beta_4)\beta_2 v_{xxx}] = 0, \tag{9a}$$

$$v_t + u_x + \epsilon[(2 + 3\beta_1\beta_4)\beta_3 v v_x + (\beta_1 + 2\beta_4)\beta_3(uv)_x + \beta_1\beta_3\beta_4 u u_x + (1 + \beta_1\beta_4)\beta_2\beta_4 v_{xxx} + (\beta_1 + \beta_4)\beta_2\beta_4 u_{xxx}] = 0 \tag{9b}$$

is integrable in the sense that it can be written as the bi-Hamiltonian system,

$$(u,v)_t^T + \theta\phi^{-1}(u,v)_x^T = 0, \tag{10}$$

where the compatible Hamiltonian operators θ and ϕ are 2×2 matrices with the entries

$$\theta_{11} = \frac{1}{1-\beta_1^2}\partial + \epsilon\beta_2\partial^3 + \epsilon\beta_3(u\partial + \partial u), \qquad \theta_{22} = \frac{1}{1-\beta_1^2}\partial + \epsilon\beta_2\beta_4^2\partial^3 + \epsilon\beta_3\beta_4(v\partial + \partial v),$$

$$\theta_{12} = -\frac{\beta_1}{1-\beta_1^2}\partial + \epsilon\beta_2\beta_4\partial^3 + \epsilon\beta_3\beta_4 u\partial + \epsilon\beta_3\partial v, \qquad \theta_{21} = -\theta_{12}^\dagger,$$

$$\phi_{11} = \phi_{22} = \frac{\beta_1}{\beta_1^2 - 1}\partial, \qquad \phi_{12} = \phi_{21} = -\frac{1}{\beta_1^2 - 1}\partial, \tag{11}$$

and † denotes the adjoint.

Proposition 3.—Let u and v solve the integrable equations (9), where $\beta_1^2 \neq 1$, $\beta_1 \neq 0$, $\beta_2 \neq 0$,

$$\beta_4^2 + 2\beta_4\beta_1 + 1 = \frac{2}{\beta_2}, \qquad \beta_4^2 + 2\frac{\beta_4}{\beta_1} + 1 = 0. \tag{12}$$

Let η and ω be defined by

$$\eta = u + \epsilon[\lambda_1 u^2 + \lambda_2 uv + \lambda_3 v^2 + \lambda_4 u_x\partial^{-1}u + \lambda_5 u_x\partial^{-1}v + \lambda_6 v_x\partial^{-1}u + \lambda_7 v_x\partial^{-1}v + \lambda_8 u_{xx} + \lambda_9 v_{xx} + x(\lambda_{10}u + \lambda_{11}v)u_x + x(\lambda_{10}v + \lambda_{11}u)v_x], \tag{13a}$$

$$\omega = v + \epsilon[\lambda_{12}v^2 + \lambda_{13}uv + \lambda_{14}u^2 + \lambda_5 v_x\partial^{-1}v + \lambda_4 v_x\partial^{-1}u + \lambda_7 u_x\partial^{-1}v + \lambda_6 u_x\partial^{-1}u + \lambda_{15}v_{xx} + \lambda_{16}u_{xx} + x(\lambda_{10}u + \lambda_{11}v)v_x + x(\lambda_{10}v + \lambda_{11}u)u_x], \tag{13b}$$

where $\lambda_1, \lambda_2, \lambda_4, \lambda_5, \lambda_8, \lambda_9$ are arbitrary, and $\lambda_3 = \lambda_1 + [3(1 + \beta_1\beta_4)\beta_3 - 1]/6$, $\lambda_6 = \lambda_5 + (\beta_1 + \beta_4)\beta_3/2$, $\lambda_7 = \lambda_4 + [3(1 + \beta_1\beta_4)\beta_3 - 1]/6$, $\lambda_{10} = [3(1 + \beta_1\beta_4)\beta_3 - 1]/2$, $\lambda_{11} = 3(\beta_1 + \beta_4)\beta_3/2$, $\lambda_{12} = (\lambda_2 - \beta_3\beta_4)/2$, $\lambda_{13} = 2\lambda_1 + \beta_3 - 1/3$, $\lambda_{14} = (\lambda_2 + \beta_1\beta_3)/2$, $\lambda_{15} = \lambda_8 + (1 + \beta_1\beta_4)\beta_2 - 1$, $\lambda_{16} = \lambda_9 + (\beta_1 + \beta_4)\beta_2$. Then η and ω solve Eqs. (1).

This result is equivalent to the following: Let $u(X,T)$ and $v(X,T)$ solve Eqs. (9) where $\beta_1^2 \neq 1$, $\beta_1 \neq 0$, $\beta_2 \neq 0$, Eqs. (12) are valid, and $X = x + \epsilon x(\lambda_{10}u + \lambda_{11}v)$, $T = t - \epsilon x(\lambda_{10}v + \lambda_{11}u)$. Let η and ω be defined by Eqs. (13) but without the x dependent terms. Then η and ω solve Eqs. (1).

Before discussing the derivation and the generalization of these results we first make some remarks.

(1) The generalized KdV equation (8) was first derived in [5] using the bi-Hamiltonian approach (see also [6]). The Lax pair of Eq. (8) and an interesting class of its solutions, called peakons, are given in [7]. The linearization of Eq. (8) using the inverse spectral method is given in [8]; if $\nu = O(1)$ the spectral theory of the equation is very similar to that of the KdV equation.

(2) The results of proposition 1 can be generalized as follows. Let $v_t + M(v) = 0$ denote the KdV, or the generalized KdV, or the generalized Gardner equation. Let v solve the equation $v_t + M(v) + \epsilon M_1(v) = 0$, where $M_1(v)$ denotes the first commuting flow of the KdV, or of the generalized KdV, or the Gardner hierarchies. Then it is possible to find a transformation of the form $\eta = v + \epsilon P(v)$, such that η solves (4). In the case of $M = K$ and $M_1 = K_1$, where K and K_1 are defined by Eqs. (2) and (5), $P(v)$ is of the form of Eq. (7), and this is precisely Kodama's result. In the other two cases $P(v)$ involves $\lambda_1 v^2 + \lambda_2 v_{xx}$ and $\lambda_1 v^2$, respectively.

(3) The case of unidirectional idealized water waves with $O(B) < O(A)$ is studied in [9]. This case is simpler than the case of $O(B) = O(A)$ since the higher dispersive terms can be neglected. It was shown by Kodama [10] that if $O(B) = O(A)$ it is impossible to extend the asymptotic integrability of the unidirectional idealized water waves [i.e., Eq. (4)] to $O(\epsilon^3)$. However, if $O(B) < O(A)$ this extension is possible [9].

(4) There exists a transformation similar to that of Eq. (6) which is valid for the Burgers equation. This transformation has been used to solve rigorously the Cauchy problem for decaying initial data of a certain physical equation describing acoustic waves [11]. It was shown in [11] that although it is possible to handle the x dependent term of the transformation using an appropriate weighted space, the rigorous analysis is much simpler if this term is absent. In this sense, the equivalent form of part (i) of proposition 1, where the x dependent term is absorbed in the t variable, is more convenient. This transformation has also been used in [12]. We emphasize that given Eq. (6) it is elementary to rewrite the transformation in an equivalent x-independent form. Indeed, $v + \epsilon x K(v) = v - \epsilon x v_t = v(x, t - \epsilon x) + O(\epsilon^2)$.

(5) The usefulness of starting with the other integrable equations of proposition 1 as opposed to KdV is questionable. A possible argument in favor of some of these equations is that their dispersion relation matches that of the full water wave, for both small and large wave numbers. The dispersion relation of the KdV, of the regularized long wave equation, of the generalized KdV, and of the full water wave equations are given by

$$-k + \epsilon k^3, \qquad -k/(1 + \epsilon k^2), \qquad -k(1 + \tfrac{3}{2}\epsilon k^2)/(1 + \tfrac{5}{2}\epsilon k^2),$$
$$-k\sqrt{1 + \epsilon\beta k^2} \Big/ \sqrt{1 + 3\epsilon\beta k^2}, \qquad \beta \text{ constant}.$$

(6) We emphasize that the results presented here are formal. In particular, it is not clear if these transformations can be used to study the initial value problem of the associated physical equations. The long wave limit of the solitary wave interaction of Eq. (4) has been studied in [13].

(7) The Kaup model [14] is a particular case of Eqs. (9) ($\beta_1 = \beta_4 = 0$). However, this model is linearly ill posed. In contrast, Eqs. (9) with general coefficients are well posed provided that $|1 + 2\beta_1\beta_4 + \beta_4^2| \geq |\beta_1 + 2\beta_4 + \beta_1\beta_4^2| > 0$.

We now discuss proposition 1 and its generalizations. Unidirectional water waves are integrable to $O(\epsilon)$. This situation arises in the asymptotic analysis of a large class of physical systems. For a subclass of such systems the $O(\epsilon)$ equation is either the KdV or the nonlinear Schrödinger equations. For systems for which the $O(\epsilon)$ equation is integrable, the concept of master symmetries provides an algorithm approach for obtaining transformations which formally extend the asymptotic integrability of these systems to $O(\epsilon^2)$ [i.e., finding transformations analogous to Eq. (6)]: *A candidate for such a transformation can be constructed from the master symmetry of the associated integrable equation by replacing the numerical coefficients in the master symmetry with arbitrary constants.* This is a consequence of the following observation: Let v solve the equation

$$v_t + M(v) + \epsilon\hat{M}_1(v) = 0. \tag{14}$$

Let u be defined by

$$u = v + \epsilon P(v). \tag{15}$$

The u solves

$$u_t + M(u) + \epsilon\hat{M}_1(u) + \epsilon[P, M]_L(u) + O(\epsilon^2) = 0, \tag{16}$$

where the commutator $[\,,\,]_L$ is defined by

$$[A, B]_L = A'B - B'A,$$

and

$$A' = \frac{\partial A}{\partial u} + \frac{\partial A}{\partial u_x}\partial_x + \frac{\partial A}{\partial u_{xx}}\partial_x^2 + \cdots.$$

Let the function $\tau(x,t)$ denote the master symmetry of the integrable equation $v_t + M(v) = 0$. This concept was introduced by one of us and Fuchssteiner [15]. The defining property of the master symmetry is that $[\tau, M]_L = M_1$, where M_1 is the next commuting flow of the associated hierarchy of integrable equations. We illustrate the importance of the master symmetry in connection with the above observation by using KdV as an example. Let $M = K$ and $\hat{M}_1 = 0$ in Eq. (14), and let $P = \tau$ in Eq. (15), where

$$\tau = \frac{1}{3}\alpha_1[8v^2 + 4v_{xx} + 2v_x\partial_x^{-1}v + x(v_{xxx} + 6vv_x)]. \tag{17}$$

Then Eq. (16) becomes $u_t + K(u) + \epsilon\alpha_1 K_1(u) = 0$, where K and K_1 are defined in Eqs. (2) and (5). The terms in K_1 differ from the $O(\epsilon)$ terms of Eq. (4) only in their numerical coefficients. Thus, in order to find the form of the transformation $P(v)$ in Eq. (6), it is natural to replace the numerical coefficients in (17) with arbitrary constants; in this way τ of Eq. (17) becomes $P(v)$ of Eq. (6).

We now discuss propositions 2 and 3 and their generalizations. The asymptotic integrability of Eqs. (1) is not immediately apparent. This situation also arises in a large class of physical systems. In order to establish the asymptotic integrability of such systems one must find the following: (i) The associated integrable system (if such a system exists). (ii) The transformation that maps this integrable system to the physical one. If question (i) can be answered, question (ii) can be approached using, as before, the master symmetry of the integrable system found in (i). There exist several methods for approaching question (i). Here we use a method based on the bi-Hamiltonian theory of integrable equations. This method is based on the following theorem [5]. Let $\theta + \lambda\phi$ be a Hamiltonian operator for all constants λ. Then the equation $u_t + \theta\phi^{-1}u_x = 0$ is an integrable equation, in the sense that it possesses infinitely many conserved quantities in involution. The bi-Hamiltonian method of finding an integrable equation associated with a given physical system consists of the following steps: (1) Seek a matrix Hamiltonian operator θ, starting with the ansatz that its components are the natural generalizations of the Hamiltonian operator of the underlying scalar integrable system. (2) Determine the relevant scalars in this ansatz by using a Maple solver to satisfy the Jacobi identity. (3) Among the Hamiltonian operators θ found in (2), choose those that give rise to a bi-Hamiltonian system.

The integrable system (9) was found by the above method: Let

$$\theta_{11} = c_1\partial + c_2\partial^3 + c_3(u\partial + \partial u) + c_4(v\partial + \partial v),$$

$$\theta_{22} = c_{11}\partial + c_{12}\partial^3 + c_{13}(u\partial + \partial u) + c_{14}(v\partial + \partial v),$$

$$\theta_{12} = -(\theta_{21})^\dagger$$
$$= c_5\partial + c_6\partial^3 + c_7u\partial + c_8\partial u + c_9v\partial + c_{10}\partial v,$$

where all the parameters appearing in these equations are constants. Demanding that the operator θ is Hamiltonian and using a Maple solver [16], one finds a set of possible constraints satisfied by the c's. One such solution is $c_{14} = c_8$, $c_9 = c_3$, $c_6 = c_2c_8/c_3$, $c_{12} = c_2c_8^2/c_3^2$, $c_4 = c_7 = c_{10} = c_{13} = 0$, $c_1, c_2, c_3, c_5, c_8, c_{11}$ arbitrary. The operator ϕ with $\phi_{11} = \hat{c}_1\partial$, $\phi_{12} = -\phi_{21}^+ = \hat{c}_5\partial$, $\phi_{22} = \hat{c}_{11}\partial$, where the $\hat{c}$'s are constants, is also Hamiltonian. Furthermore, since c_1, c_5, c_{11} are arbitrary, $\theta + \lambda\phi$ is also Hamiltonian. Thus the system of equations $(u,v)_t^T + \theta A(u,v)^T = 0$, where the components of the matrix A are given by $A_{11} = \sigma_1$, $A_{12} = A_{21} = 1$, $A_{22} = \sigma_2$, σ_1, σ_2 arbitrary constants, is integrable. Demanding that the coefficients of v_x and u_x in the first and the second equations are $1, 0$ and $0, 1$, respectively, it follows that $\sigma_1 = \sigma_2$, $c_5 = -c_1\sigma_1$, $c_1 = c_{11} = 1/(1 - \sigma_1^2)$. Renaming $\sigma_1 = \beta_1$, $c_8/c_3 = \beta_4$, $c_2 = \epsilon\beta_2$, $c_3 = \epsilon\beta_3$ these equations become Eqs. (9).

We now discuss proposition 3. The form of the transformation (13) follows from the form of the master symmetry of Eqs. (9). It is straightforward to show that if u, v solve

$$u_t + v_x + \epsilon[\tilde{\alpha}_1uu_x + \tilde{\alpha}_2uv_x + \tilde{\alpha}_3u_xv + \tilde{\alpha}_4vv_x + \tilde{\alpha}_5u_{xxx} + \tilde{\alpha}_6v_{xxx}] = 0,$$

$$v_t + u_x + \epsilon[\tilde{\alpha}_7vv_x + \tilde{\alpha}_8u_xv + \tilde{\alpha}_9uv_x + \tilde{\alpha}_{10}uu_x + \tilde{\alpha}_{11}v_{xxx} + \tilde{\alpha}_{12}u_{xxx}] = 0,$$

and if η and ω are defined by Eqs. (13), then η, ω solve

$$\eta_t + \omega_x + \epsilon[\alpha_1\eta\eta_x + \alpha_2\eta\omega_x + \alpha_3\eta_x\omega + \alpha_4\omega\omega_x + \alpha_5\eta_{xxx} + \alpha_6\omega_{xxx}] + O(\epsilon^2) = 0,$$

$$\omega_t + \eta_x + \epsilon[\alpha_7\omega\omega_x + \alpha_8\eta_x\omega + \alpha_9\eta\omega_x + \alpha_{10}\eta\eta_x + \alpha_{11}\omega_{xxx} + \alpha_{12}\eta_{xxx}] + O(\epsilon^2) = 0,$$

where $\lambda_1, \lambda_2, \lambda_4, \lambda_5, \lambda_8, \lambda_9$ are arbitrary, $\tilde{\alpha}_5 + \tilde{\alpha}_{11} = \alpha_5 + \alpha_{11}$, $\tilde{\alpha}_6 + \tilde{\alpha}_{12} = \alpha_6 + \alpha_{12}$, $\lambda_3 = \lambda_1 + (\hat{\alpha}_2 - \hat{\alpha}_3 + \hat{\alpha}_7 - \hat{\alpha}_{10})/4$, $\lambda_6 = \lambda_5 + (\hat{\alpha}_1 - \hat{\alpha}_4 - \hat{\alpha}_8 + \hat{\alpha}_9)/4$, $\lambda_7 = \lambda_4 + (-\hat{\alpha}_2 + \hat{\alpha}_3 + \hat{\alpha}_7 - \hat{\alpha}_{10})/4$, $\lambda_{10} = (\hat{\alpha}_2 + \hat{\alpha}_3 + \hat{\alpha}_7 + \hat{\alpha}_{10})/4$, $\lambda_{11} = (\hat{\alpha}_1 + \hat{\alpha}_4 + \hat{\alpha}_8 + \hat{\alpha}_9)/4$, $\lambda_{12} = \lambda_2/2 + (\hat{\alpha}_4 - \hat{\alpha}_8)/4$, $\lambda_{13} = 2\lambda_1 + (\hat{\alpha}_2 - \hat{\alpha}_{10})/2$, $\lambda_{14} = \lambda_2/2 + (\hat{\alpha}_1 - \hat{\alpha}_9)/4$, $\lambda_{15} = \lambda_8 + \hat{\alpha}_6$, $\lambda_{16} = \lambda_9 + \hat{\alpha}_5$, and $\hat{\alpha}_j$ are defined by $\hat{\alpha}_j = \tilde{\alpha}_j - \alpha_j$, $j = 1, \ldots, 12$. Proposition 3 follows from the above and the fact that in our particular case u and v satisfy Eqs. (9), while η and ω satisfy Eqs. (1). The equivalent formulation follows from the observation that xv_x and xu_x in (13a) and (13b) can be replaced by $-xu_t$ and $-xv_t$, respectively, and that $u + \epsilon x(\lambda_{10}u + \lambda_{11}v)u_x - \epsilon x(\lambda_{10}v + \lambda_{11}u)u_t = u(x + \epsilon x(\lambda_{10}u + \lambda_{11}v), t - \epsilon x(\lambda_{10}v + \lambda_{11}u)) + O(\epsilon^2)$.

This work was partially supported by the National Science Foundation DMS-91116111, by the Air Force

Office of Scientific Research F49620-93-1-0088, and by the EPSRC GR1571885.

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