Gaussian Pulse Propagation in a Dispersive, Absorbing Dielectric

Kurt E. Oughstun
College of Engineering & Mathematics, University of Vermont, Burlington, Vermont 05405

Constantinos M. Balictsis
Department of Computer Science & Electrical Engineering, University of Vermont, Burlington, Vermont 05405
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The modified asymptotic description of dispersive Gaussian pulse propagation, which is uniformly valid in the initial pulse envelope width, is shown to reduce to the energy velocity description when the propagation distance becomes sufficiently large in a Lorentz model dielectric. This then resolves the apparent controversy between the modern asymptotic description upon which the energy velocity description is based and the classical group velocity description of Gaussian pulse propagation and related experimental results. [S0031-9007(96)01070-8]

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The effects of frequency dispersion and absorption on the dynamical evolution of an electromagnetic pulse as it propagates through a homogeneous, isotropic, linear dielectric are properly described by asymptotic methods of analysis as originally investigated by Sommerfeld [1] and Brillouin [2] in 1914 using the method of steepest descents and improved upon and corrected by Oughstun and Sherman [3–5] using modern asymptotic expansion techniques [5]. This analysis clearly shows that after the pulse has propagated a sufficiently large distance into the medium its dynamics settle into a mature dispersion regime [5,6] in which the propagated field becomes locally quasimonochromatic with fixed local frequency, wavelength, and attenuation in each region of space that travels with its own characteristic velocity. The theory provides asymptotic expressions for the local wave properties at any given space-time point in the field domain. A physical explanation of these local wave properties was provided by Sherman and Oughstun [7] in 1981 with its entire proof just recently given [8]. In 1982, Chu and Wong [9] published experimental results for picosecond laser pulses propagating through thin samples of a linear dispersive dielectric whose peak absorption never exceeded 6 absorption lengths that were purported to disprove the energy velocity description while verifying the group velocity description. When combined with the Gaussian pulse results [10] of the modern asymptotic theory, the recently published modified asymptotic description [11,12] of Gaussian pulse propagation is found to provide the basis for a complete explanation of this apparent discrepancy.

Consider an input Gaussian envelope modulated harmonic wave of constant applied carrier frequency $\omega_c > 0$ and initial full pulse width $2T > 0$ that is centered about the time $t_0 > 0$ at the plane $z = 0$, given by

$$\begin{aligned} f(t) &= u(t)\sin(\omega_c t + \psi) \\ &= \exp\left\{-\left(\frac{t - t_0}{T}\right)^2\right\}\sin(\omega_c t + \psi)\,, \end{aligned} \tag{1}$$

which is propagating in the positive z direction through a linear dielectric whose frequency dispersion is described by the single resonance Lorentz model with complex index of refraction

$$n(\omega) = \left(1 - \frac{b^2}{\omega^2 - \omega_0^2 + 2i\delta\omega}\right)^{1/2}, \tag{2}$$

which occupies the source-free half space $z \geq 0$. Here ω_0 is the undamped resonance frequency, b is the plasma frequency, and δ is the phenomenological damping constant of the dispersive, lossy dielectric. The integral representation of the propagated plane wave pulse in the half space $z \geq 0$ is given by

$$A(z,t) = \frac{1}{2\pi}\int_C \tilde{f}(\omega)\exp\left[\frac{z}{c}\,\phi(\omega,\theta)\right]d\omega\,, \tag{3}$$

where $\theta = ct/z$ is a dimensionless space-time parameter,

$$\phi(\omega,\theta) = i\omega[n(\omega) - \theta] \tag{4}$$

is the classical complex phase function, and where $\tilde{f}(\omega)$ is the temporal Fourier spectrum of the initial pulse $f(t) = A(0,t)$ at the input plane at $z = 0$. The spectral amplitude $\tilde{A}(z,\omega)$ of $A(z,t)$ satisfies the scalar Helmholtz equation $[\nabla^2 + \tilde{k}^2(\omega)]\tilde{A}(z,\omega) = 0$, with complex wave number $\tilde{k}(\omega) = \omega n(\omega)/c$.

From Eqs. (1) and (3), the classical integral representation of the propagated Gaussian envelope pulse is found as

$$A(z,t) = \frac{1}{2\pi}\Re\left\{i\int_C \tilde{u}(\omega - \omega_c)\exp\left[\frac{z}{c}\,\phi(\omega,\theta')\right]d\omega\right\}, \tag{5}$$

for $z \geq 0$, with the initial pulse spectrum

$$\tilde{u}(\omega) = \pi^{1/2}T\exp\left[-\frac{T^2}{4}\,\omega^2\right]\exp[-i(\omega_c t_0 + \psi)]\,, \tag{6}$$

where $\theta' = \theta - ct_0/z$. The contour of integration C appearing here may be taken as any contour in the complex ω plane that is homotopic to the real frequency

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axis. Since this spectrum is an entire function of ω, the propagated field has the asymptotic representation [10]

$$A(z,t) \sim A_S(z,t) + A_B(z,t), \tag{7}$$

as $z \rightarrow \infty$, with

$$A_j(z,t) = a_j \left(\frac{c}{2\pi z}\right)^{1/2} \Re\left\{ i \, \frac{\tilde{u}(\omega_{SP_j} - \omega_c)}{[-\phi^{(2)}(\omega_{SP_j}, \theta')]^{1/2}} \times \exp\left[\frac{z}{c}\, \phi(\omega_{SP_j}, \theta')\right]\right\} \tag{8}$$

for $j = S, B$. Here $a_S = 2$ and $\omega_{SP_j} = \omega_{SP_D}^{+}(\theta')$ denotes the distant first-order saddle point location of $\phi(\omega, \theta')$ in the right half of the complex ω plane for all $\theta' > 1$, while $a_B = 1$ for $1 < \theta' < \theta_1$ and $a_B = 2$ for $\theta_1 < \theta'$ where $\omega_{SP_B} = \omega_{SP_N}^{+}(\theta')$ denotes the near first-order saddle point location of $\phi(\omega, \theta')$ in the right half of the complex ω plane. Here $\theta_1 \cong \theta_0 + 2\delta^2 b^2/\theta_0 \omega_0^2 (3\omega_0^2 - 4\delta^2)$ denotes the space-time point at which the two near first-order saddle points coalesce into a single second-order saddle point, where $\theta_0 = n(0) = (1 + b^2/\omega_0^2)^{1/2}$ denotes the space-time point at which the upper near saddle point crosses the origin [3–5]. The nonuniform behavior exhibited in Eqs. (7) and (8) in any small neighborhood of the space-time point $\theta' = \theta_1$ may be corrected using uniform asymptotic expansion techniques [4,5]. The asymptotic contribution due to the near saddle points is referred to as a generalized Brillouin precursor field, while that due to the distant saddle points is referred to as a generalized Sommerfeld precursor field [10,11].

Because of the initial Gaussian envelope spectrum (6), the asymptotic description of each pulse component $A_S(z,t)$ and $A_B(z,t)$ contains a Gaussian amplitude factor of the form $\exp\{-(T/2)^2[\Re(\omega_{SP_j}) - \omega_c]^2\}$, $j = S, B$. In addition, each pulse component contains an exponential attenuation factor that is given by the product of the propagation distance z with the attenuation that is characteristic of the real phase behavior $\Re\{\phi(\omega_{SP_j})\}$ at the relevant saddle point, and the instantaneous oscillation frequency of each pulse component in the mature dispersion regime is approximately given by $\Re\{\omega_{SP_j}\}$ in the ultrashort pulse limit as $T \rightarrow 0$. Consequently, for a below resonance carrier frequency $\omega_c \in (0, \omega_0)$ the instantaneous oscillation frequency of the generalized Brillouin precursor $A_B(z,t)$ crosses ω_c as it chirps upward towards ω_0, while for an above resonance carrier frequency $\omega_c \in (\omega_1, \infty)$ the instantaneous oscillation frequency of the generalized Sommerfeld precursor $A_S(z,t)$ crosses ω_c as it chirps downwards towards ω_1, in each case the Gaussian amplitude factor peaking to unity when $\Re\{\omega_{SP_j}(\theta')\} = \omega_c$. For an intra-absorption band carrier frequency $\omega_c \in (\omega_0, \omega_1)$ the carrier frequency is never attained by either pulse component.

If the input signal frequency ω_c is within the absorption band of the medium, so that $\omega_0 \leq \omega_c \leq \omega_1$, then both pulse components $A_S(z,t)$ and $A_B(z,t)$ will be present in roughly equal proportion; the Brillion precursor component $A_B(z,t)$ becomes more pronounced as ω_c is decreased from ω_1 to ω_0 and dominates the propagated field evolution as ω_c is decreased below the medium resonance frequency, whereas the Sommerfeld precursor component $A_S(z,t)$ becomes more pronounced as ω_c is increased from ω_0 to ω_1 and dominates the propagated field evolution as ω_c is increased above ω_1. The numerically determined dynamical field evolution, due to an input ultrashort Gaussian pulse with initial pulse width $2T = 0.2$ fsec and carrier frequency $\omega_c = 5.75 \times 10^{16}$ sec^{-1} that is near the upper end of the absorption band of a single resonance Lorentz medium with parameters $\omega_0 = 4 \times 10^{16}$ sec^{-1}, $b^2 = 20 \times 10^{32}$ sec^{-2}, and $\delta = 0.28 \times 10^{16}$ sec^{-1}, is illustrated in Fig. 1. This case is of particular interest since the group velocity at this signal frequency is very nearly equal to the speed of light c in vacuum. The generalized Sommerfeld precursor pulse component is seen to first emerge in the propagated field structure as the propagation distance increases into the mature dispersion regime, its peak amplitude propagating with a velocity just below c; notice that the smallest propagation distance considered is nearly 21 absorption depths into the medium at this intra-absorption band carrier frequency. As the propagation distance increases, the generalized Brillouin precursor pulse component emerges, its peak amplitude propagating with a velocity that approaches the value $c/\theta_0 = c/n(0)$ from above. The propagated field due to an input ultrashort Gaussian pulse then separates into two distinct pulse components that propagate with different peak velocities, the faster pulse component being the high-frequency generalized Sommerfeld precursor whose instantaneous oscillation frequency $\omega_s(\theta)$ chirps downward towards ω_1, followed by the slower, low-frequency generalized Brillouin precursor whose instantaneous oscillation frequency $\omega_B(\theta)$ chirps upward towards ω_0. Each feature of this dynamical field evolution is properly described by the energy velocity description of Refs. [7,8].

As the initial pulse width $2T$ is increased, the asymptotic approximation (7) and (8) of the propagated field evolution remains qualitatively correct, while its quantitative accuracy decreases at a fixed propagation distance. This asymptotic description will remain quantitatively accurate as the pulse width is increased provided that the propagation distance is allowed to increase, in keeping with the definition of an asymptotic expansion in Poincare's sense [13] as $z \rightarrow \infty$. However, since the medium is attenuative, the usefulness of this description decreases as $2T$ increases, since the important features of the field evolution (particularly when compared to experimental observations) are typically observed at some fixed observation distance in the medium.

The classical integral representation (5) with the spectrum (6) may be rearranged so as to yield the modified integral representation [12]

$$A(z,t) = \frac{1}{2\pi} \Re\left\{ i \int_C \tilde{U}_M \exp\left[\frac{z}{c}\, \Phi_M(\omega, \theta')\right] d\omega \right\} \tag{9}$$

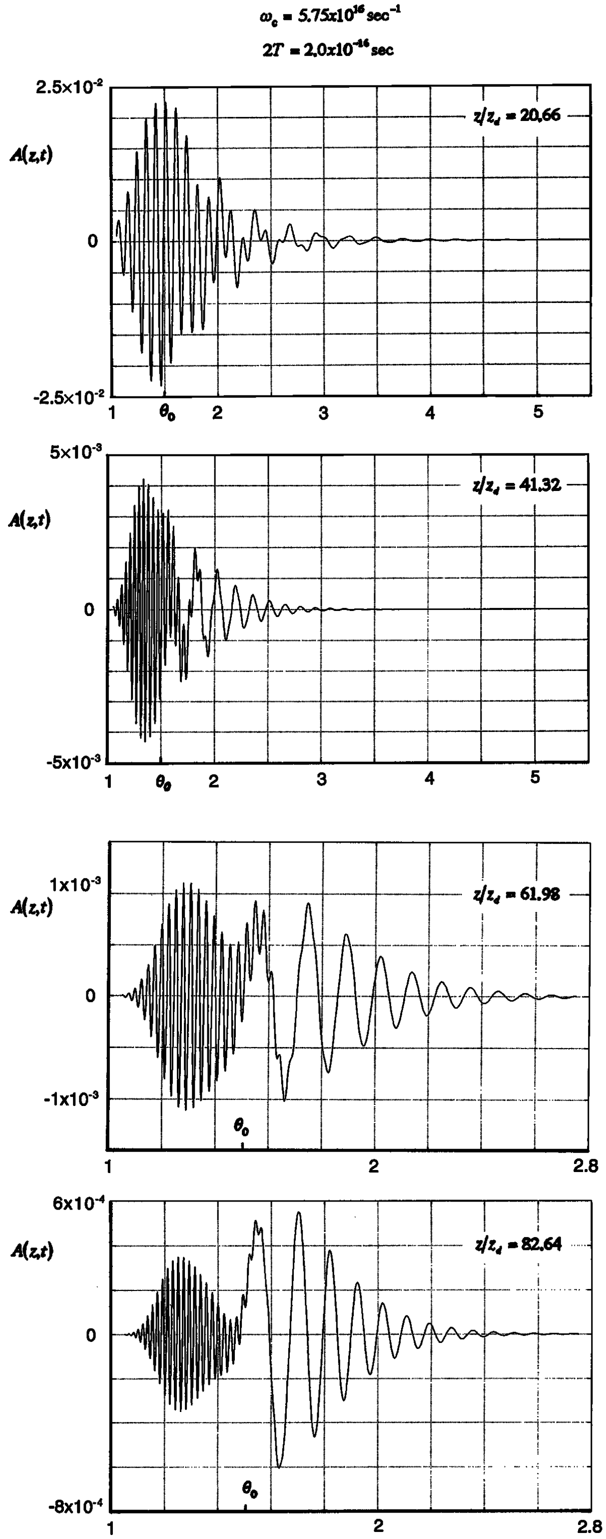

FIG. 1. Numerically determined dynamical field evolution of an input 0.2 fsec Gaussian pulse with intra-absorption band carrier frequency $\omega_c = 5.75 \times 10^{16}\ \text{sec}^{-1}$ in a single resonance Lorentz model dielectric.

for all $z \geq 0$, where the modified spectral amplitude function

$$\tilde{U}_M = \pi^{1/2} T \exp[-i(\omega_c t_0 + \psi)] \tag{10}$$

is independent of the angular frequency ω, and where

$$\Phi_M(\omega, \theta') = \phi(\omega, \theta') - \frac{cT^2}{4z} (\omega - \omega_c)^2 \tag{11}$$

is the modified complex phase function. In the ultrashort pulse limit, as $2T \to 0$, the modified phase function reduces to the classical phase function $\phi(\omega, \theta')$ and the asymptotic behavior of (9) is determined by the behavior about the saddle points of $\phi(\omega, \theta')$, as in Eqs. (7) and (8). Hence, *if the classical asymptotic description given in Eqs. (7) and (8) is valid (to some specific degree of accuracy) for some given input pulse width 2T at a given propagation distance z, then this description will remain equally valid (to that same degree of accuracy) as the initial pulse width is increased provided that z is also increased in such a manner that the ratio T^2/z remains fixed.*

The saddle point dynamics of the modified phase function are now dependent upon both the initial pulse width and the propagation distance, as well as upon the dimensionless space-time parameter θ'. These saddle points are found [12] to remain isolated from each other for all θ' when $T \neq 0$ and are each of first order. Only two of these saddle points are found [12] to contribute to the asymptotic behavior of the modified integral representation (9) as $z \to \infty$, so that the propagated field has the same asymptotic representation given in Eq. (7) with

$$A_j(z,t) = \left(\frac{c}{2\pi z}\right)^{1/2} \Re\left\{ i \frac{\tilde{U}_M}{[-\Phi_M^{(2)}(\omega_j, \theta')]^{1/2}} \times \exp\left[\frac{z}{c} \Phi_M(\omega_j, \theta')\right]\right\} \tag{12}$$

for $j = S, B$. Here ω_j denotes the modified distant ($j = S$) and near ($j = B$) saddle-point locations in the right half of the complex ω plane whose dynamics are described in Ref. [12]. Each pulse component $A_j(z,t)$, $j = S, B$, contains a Gaussian amplitude factor, the peak amplitude point of each pulse component propagating at the classical group velocity evaluated at the instantaneous oscillation frequency of the field at the space-time point.

The dispersive action of the same Lorentz model dielectric on an input 5 fsec Gaussian pulse, whose carrier frequency $\omega_c = 5.625 \times 10^{16}\ \text{sec}^{-1}$ is just below the upper end of the medium absorption band, produces a superluminal velocity of the peak in the envelope of the propagated field at a sufficiently small propagation distance, as indicated in Fig. 2 by data point 1. The envelope peak in the propagated field at this propagation distance (19.96 absorption depths at ω_c) has the associated instantaneous frequency $\omega_{p_s} \cong 5.71 \times 10^{16}\ \text{sec}^{-1}$, and it propagates with the classical group velocity $v_{p_s} = v_g(\omega_{p_s}) \cong 1.16c$. This same envelope peak slows down to a subluminal velocity as the propagation distance increases (data point 2) because the instantaneous oscillation frequency of this peak amplitude point increases as the propagation distance increases. The envelope peak in the propagated field at this propagation distance (49.91 absorption depths at ω_c)

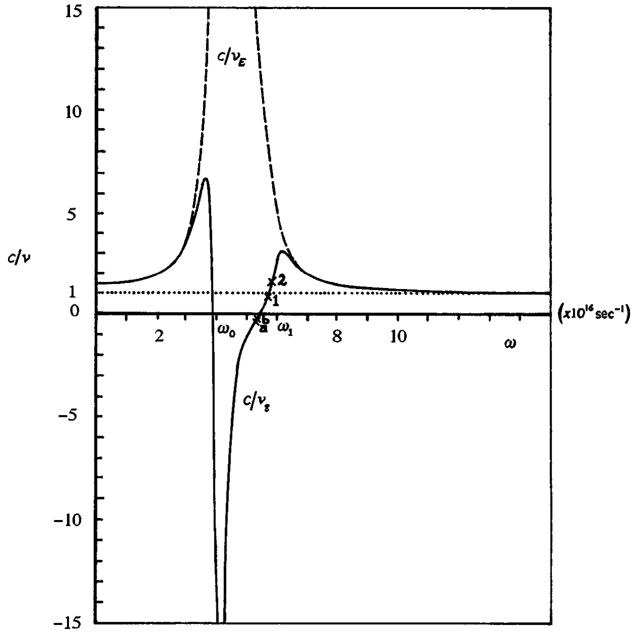

FIG. 2. Inverse relative velocity of propagation of the peak amplitude point of the propagated field due to an input Gaussian pulse (data points 1,2 and a, b). The solid curve describes the frequency dependence of the inverse relative group velocity $c/v_g(\omega_{p_k})$ evaluated at the instantaneous oscillation frequency ω_{p_k} of the peak amplitude point of the propagated pulse component $A_k(z,t)$, while the dashed curve in the figure describes the frequency dependence of the inverse relative energy velocity $c/v_E(\omega_{p_k})$ of a monochromatic field of angular frequency ω_{p_k}.

has shifted to the higher instantaneous oscillation frequency $\omega_{p_s} \cong 5.83 \times 10^{16}$ sec^{-1}, and it now propagates with the classical group velocity $v_{p_s} = v_g(\omega_{p_s}) \cong 0.65c$. Thus, as the propagation distance increases, the instantaneous oscillation frequency evolves out of the absorption band and the pulse dynamics evolve toward the energy velocity description which is valid in the mature dispersion regime.

Negative velocity motions of the amplitude peak are obtained from the modified asymptotic description [12] for an input 10 fsec Gaussian pulse with applied carrier frequency $\omega_c = 5.25 \times 10^{16}$ sec^{-1}, as indicated by data points a and b in Fig. 2. At the smallest propagation distance considered (58.05 absorption depths at ω_c) the envelope peak of the propagated pulse has the associated instantaneous oscillation frequency $\omega_{p_s} \cong 5.29 \times 10^{16}$ sec$^{-1} > \omega_c$ and propagates with the classical group velocity $v_{p_s} = v_g(\omega_{p_s}) \cong -2.86c$. As the propagation distance is increased to 145.13 absorption depths, the instantaneous oscillation frequency at the envelope peak has shifted to the higher frequency value $\omega_{p_s} \cong 5.35 \times 10^{16}$ sec^{-1} and the envelope peak now propagates with the classical group velocity $v_{p_s} = v_g(\omega_{p_s}) \cong -4.45c$. The modified asymptotic description then shows that, as the propagation distance increases, the low-frequency components that are present in the input pulse spectrum are attenuated at a larger rate than are the high-frequency components, so that the propagated pulse spectrum becomes dominated by an increasingly higher frequency component, and the peak in the envelope of the propagated pulse propagates with the group velocity at this frequency value. Again, as the propagation distance increases into the mature dispersion regime, the pulse dynamics evolve toward the energy velocity description; however, the overall field amplitude also rapidly attenuates to zero in this case.

Because of the small propagation distance of at most 6 absorption depths in their laboratory arrangement, the experimental results of Chu and Wong [9] are restricted to the small propagation distance limit below the mature dispersion regime. The modified asymptotic description [12] bridges the gap between these two regimes, being in agreement with the experiment results [9] at small propagation distances, while reducing to the classical asymptotic description at sufficiently large propagation distances in the dispersive, lossy medium. Moreover, the modified asymptotic description provides, for the first time, a mathematically rigorous derivation of the correct group velocity description of Gaussian pulse propagation in a dispersive, lossy medium and clearly shows how that description evolves into the energy velocity description as the propagation distance increases into the mature dispersion regime.

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