Synchronizing Spatiotemporal Chaos in Coupled Nonlinear Oscillators

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The synchronization of spatiotemporal chaos of two arrays of coupled nonlinear oscillators is achieved by discrete time coupling of individual cells of the arrays. This synchronization method is based on the knowledge of the local dynamics and can be applied to any type of arrays where the synchronization properties of the cells are known. Furthermore, we discuss possible applications of synchronizing spatiotemporal chaos in communication and anticontrol of chaos. [S0031-9007(96)00935-0]

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Many nonlinear phenomena in physics, biology, and engineering can be modeled by an array of diffusively coupled oscillators or, in other words, coupled ordinary differential equations (CODE). A list of examples of such phenomena includes: current-biased series arrays of Josephson junctions [1], dynamic arrays of nonlinear electrical circuits [2], discrete reaction-diffusion equations [3], and networks of neurons and cardiac pacemaker cells [4]. The most interesting type of behavior encountered in large interconnections of chaotic oscillators is spatiotemporal chaos where the observed dynamics exhibits chaotic properties both in time and space. In this Letter we describe a general method for synchronizing *pairs* of unidirectionally coupled CODEs with spatiotemporal chaotic dynamics. Chaos synchronization [5–7], generalized synchronization [8,9] and phase synchronization [10,11], in dynamical systems are presently a field of active research in view of potential applications in communication [12–14] and system identification [15,16].

Chaos synchronization *within* single CODEs has also been studied extensively. For example, synchronized behavior has been investigated in mean-field coupled Lorenz oscillators [17], Rössler oscillators [18], laser systems [19], neural networks [20], and electronic circuits [21]. In all these examples, synchronization leads to a coherent motion of the elements within the given array, that is, the variables in the ith and jth cell are identical in time. However, the problem of interaction between many CODEs, and especially their synchronization [22], is important for both understanding the nature of CODEs and potential applications.

This Letter is organized as follows. First we illustrate numerically how local coupling at discrete times leads to the synchronization of spatiotemporal chaos in an array of diffusively coupled Lorenz systems. Then we give arguments why it can be expected that the coupling mechanism used leads to synchronization for a large class of pairs of CODEs, and finally we discuss the relevance of our results for applications in communication systems and anticontrol of chaos.

To demonstrate spatiotemporal synchronization of arrays, we use as an example an array of N diffusively coupled Lorenz systems

$$\begin{aligned} \dot{x}_i &= \sigma(y_i - x_i) + D(x_{i+1} - 2x_i + x_{i-1}), \\ \dot{y}_i &= rx_i - y_i - x_i z_i, \\ \dot{z}_i &= x_i y_i - bz_i, \end{aligned} \tag{1}$$

where $i = 1, \ldots, N$, and the values of the parameters are fixed to $\sigma = 10$, $r = 23$, $b = 1$, and $D = 6$. For the numerical simulations presented in this Letter, we use periodic boundary conditions $x_0 = x_N$ and $x_{N+1} = x_1$, but similar results have been obtained with other types of boundary conditions. Figure 1(a) depicts the spatiotemporal evolution of an array of $N = 100$ Lorenz systems. This figure shows the grey-coded values of the x coordinates as a function of time t and spatial coordinate i. The evolution is chaotic, with Lyapunov dimension equal to $D_L = 69.3$. The array (1) drives a similar array

$$\begin{aligned} \dot{\tilde{x}}_i &= \sigma(\tilde{y}_i - \tilde{x}_i) + D(\tilde{x}_{i+1} - 2\tilde{x}_i + \tilde{x}_{i-1}), \\ \dot{\tilde{y}}_i &= r\tilde{x}_i - \tilde{y}_i - \tilde{x}_i \tilde{z}_i, \\ \dot{\tilde{z}}_i &= \tilde{x}_i \tilde{y}_i - b\tilde{z}_i, \end{aligned} \tag{2}$$

which will be called *response system* in the following. The coupling between the two arrays (1) and (2) is active at discrete times only. This type of *sporadic coupling* of continuous systems has been introduced for low dimensional systems only recently [23,24]. The dynamics of the jth cell in (2) is influenced by the *drive system* (1) in the following way. At the moment $t_{j,n} = jT_1 + (n - 1)NT_2$, $n = 1, 2, \ldots$, the state variable $\tilde{y}_j$ is replaced by the value of the state variable y_j, that is $\tilde{y}_j(t_{j,n}) = y_j(t_{j,n})$ [25]. In other words, the dynamical system (2) oscillates freely and independently from the drive system (1) except for the moments $t_{1,n}, t_{2,n}, \ldots, t_{N,n}$ when the variables $\tilde{y}_{1,n}, \tilde{y}_{2,n}, \ldots, \tilde{y}_{N,n}$ are forced to the new values $y_{1,n}, y_{2,n}, \ldots, \tilde{y}_{N,n}$, respectively [26]. In general $t_{j,n} < t_{(j+1),n}$, although the case $T_1 = 0$, that is $t_{j,n} = t_{(j+1),n}$ for all j, can also be considered (simultaneously driving of cells in

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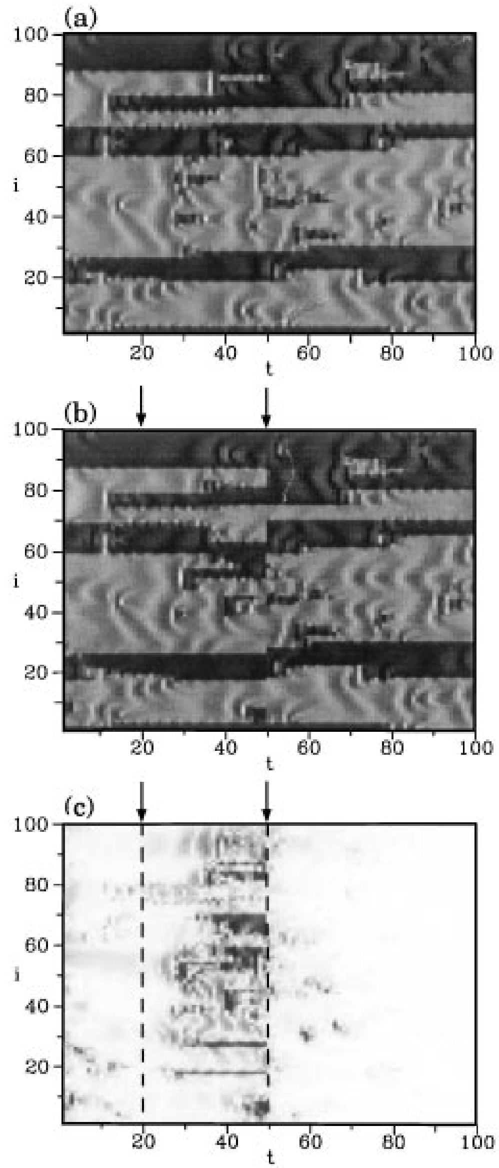

FIG. 1. Synchronization of spatiotemporal chaos. (a) Spatiotemporal evolution of the x coordinates of the array (1) as a function of time t and spatial coordinate i. (b) The same as in (a) for the response array (2). The arrows denote the times $t = 20$ and $t = 50$ when the coupling between the arrays is switched off and on, respectively. (c) The same as in (a) for the difference $|x - \tilde{x}|$.

the response system). For simplicity, in the following we use $T_1 = T_2 = T$.

The results of our numerical simulations are shown in Fig. 1. Figure 1(b) depicts the spatiotemporal evolution of the array (2). In the simulation we have used $T = 0.01$. This means that the time interval $t_{j+1,n} - t_{j,n}$ between the driving of neighboring cells equals $T = 0.01$ and that a fixed Lorenz system in the response systems is coupled to its counterpart in the drive array after time intervals of length NT, which in our case is equal to 1. To show how effective this method of synchronization is, we switch off the coupling between the two systems at time $t = 20$ and switch it on again at $t = 50$ as denoted by the arrows. One can see that the synchronization is achieved already after a short time of 30 time units.

Figure 1(c) shows the modulus of the difference of the x variables of the drive and the response system. The dark regions indicate the desynchronization of the arrays in particular during the time interval $20 < t < 50$ when the coupling is switched off. This effect can also be seen in Fig. 2 that gives the global synchronization error e as a function of time, whereby

$$e = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \tilde{x}_i)^2 + (y_i - \tilde{y}_i)^2 + (z_i - \tilde{z}_i)^2}\,. \tag{3}$$

As can be seen the synchronization error tends to zero as soon as the coupling is switched on.

We explain now why the two arrays synchronize despite the fact that most of the time they oscillate freely and independently. For this purpose we first consider the synchronization mechanism for a pair of individual cells, i.e., without the diffusive coupling to neighboring cells within the array. A single cell of the drive

$$\begin{aligned} \dot{x} &= \sigma(y - x)\,, \\ \dot{y} &= rx - y - xz\,, \\ \dot{z} &= xy - bz\,, \end{aligned} \tag{4}$$

drives a cell of the response

$$\begin{aligned} \dot{\tilde{x}} &= \sigma(\tilde{y} - \tilde{x})\,, \\ \dot{\tilde{y}} &= r\tilde{x} - \tilde{y} - \tilde{x}\tilde{z}\,, \\ \dot{\tilde{z}} &= \tilde{x}\tilde{y} - b\tilde{z}\,, \end{aligned} \tag{5}$$

in the following way: At discrete times $t_n = n\tau$, $n = 1, 2, \ldots$, the value of the $\tilde{y}$ component is forced to the new value y, that is $\tilde{y}(t_n) = y(t_n)$. Let us assume for a moment that this unidirectional driving is applied continuously as time goes on. In this case $\tilde{y}(t) = y(t)$ for all t, and Eq. (4) becomes the (x, z) subsystem of the Lorenz system, which is known to be asymptotically stable [7]. This property ensures synchronization of

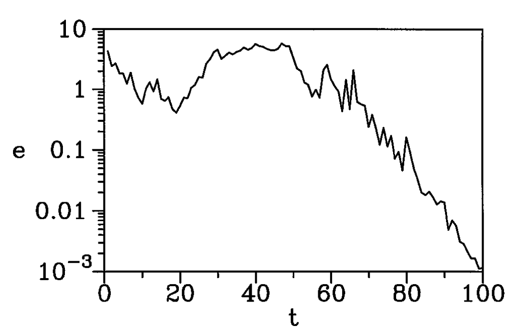

FIG. 2. The synchronization error Eq. (3) versus time.

drive and response even when they are coupled only at discrete times [23,24,27]. Indeed, using the analytical arguments of [23], from asymptotical stability of the (x, z) subsystem, it follows that there exists a critical value τ_c such that for all $\tau < \tau_c$, sporadically coupled systems (4) and (5) are synchronized. Numerically we have found that in this case $\tau_c = 0.45$.

Now let us consider the arrays (1) and (2) again. Our numerical simulations for different values of the parameters of the Lorenz system, the coupling constant D and the characteristic time for sporadic coupling T, show that there exist critical values T_c and D_c such that for all $T < T_c$ and $D < D_c$, the arrays (1) and (2) are synchronized [28]. We give now a heuristic explanation of this property. If the coupling D is small, then the cells behave almost independently, and on the basis of the analysis for a single cell, one can conclude that the arrays will synchronize. Clearly, in this case $T_c \approx \tau_c/N = 0.0045$, because $T_c N = \tau_c = 0.45$. On the other hand, if the coupling D is large, for example $D = 6$ as above, then the dissipative character of the coupling tends to diminish the differences between cells in each array, and together with the arguments given above for a single cell, this will also lead to a synchronization of both arrays. In general, scalar diffusive coupling can also lead to synchronization of the cells *within* a given array. In such cases the dynamics of the whole array is reduced to the dynamics of a single cell, and the array will thus not show spatiotemporal chaos. The graphical illustration in Fig. 1(a) and the high Lyapunov dimension of $D_L = 69.3$, however, show that the array (1) is for $D = 6$ not in a synchronous state. Nevertheless, the dissipative character of the coupling influences strongly the values of critical parameters T_c and D_c. This is the reason why the cells are synchronized even for $\tau = TN = 1 > \tau_c = 0.45$. That this interpretation of the observed phenomena of synchronization is correct is also supported by the fact that we were able to synchronize the two arrays (1) and (2) even when they are coupled every second cell only. This property is an immediate consequence of the diffusive type of interaction within a single array. Therefore the synchronization of the arrays (1) and (2) is thus due to two facts: (i) the dissipative character of the interaction between the cells in the arrays and (ii) the synchronization properties of the sporadically driven cells. We emphasize here that the knowledge about the local dynamics (of each cell) is used to manipulate systematically the behavior of the global dynamics of the whole CODE, namely to synchronize two CODEs.

How general is this method for synchronization of spatiotemporal chaos? We have performed numerical experiments with different arrays of coupled (non)identical cells and different types of internal couplings between cells within the arrays. In all cases we have observed similar results, namely, if the sporadic coupling between single cells leads to an asymptotically stable subsystem in the limit of time continuous coupling, then there exist critical values for the coupling constant in the array and the characteristic time of sporadic coupling [29] such that two arrays coupled at discrete times are synchronized. A more detailed analysis of the above results, as well as some generalizations for the case of synchronization based on active-passive decomposition [14], will be presented in an extended version of this work. We stress here that the method is applicable not only to arrays, but also in the case of PDEs.

We now discuss two possible applications of the synchronization method introduced in this Letter.

The first application is communication. The basic idea is to transmit an information signal using a chaotic signal as a broadband carrier and the synchronization is necessary to recover the information at the receiver. This idea was discussed, for example, in [13,14]. Assume that the drive system is the transmitter, its copy is the receiver, and they are u-sporadically coupled, where u stands for a variable of the cells in the array [in the example above, (1) is the transmitter, (2) is the receiver, and $u = y$]. Analogous to the encoding method proposed in Refs. [13,14] the information signal is injected in the transmitter, while the transmitted signal is a sequence of numbers: $u_1(T)$, $u_2(T), \dots, u_N(T)$, $u_1(2T)$, $u_2(2T), \dots, u_N(2T), \dots$. Using similar arguments as in [13,14], one can show that the information can in principle be recovered at the receiver without errors. The advantage of the new implementation using sporadic coupling is the fact that the transmitted signal is only a discretely sampled sequence of numbers and that this sequence is generated by a hyperchaotic system. We would like to point out here the recent question of synchronizing hyperchaotic systems with a scalar continuous signal. That this is possible was shown, for example, in [14,31,32] for a *continuous* driving signal. Moreover, with the sporadic driving used in this Letter it is also possible with a scalar *discretely* sampled signal.

The second application is anticontrol and control of chaos. It is well known that in some biomedical systems, chaotic behavior is "normal," or, in another words, the loss of chaos in these systems is often associated with disease. A list of examples of emergent regular (periodic) behavior in otherwise irregular (chaotic) but normal dynamics includes: cardiac interbeat interval patterns in a variety of cardiac disorders [33], experimental epilepsy [34], immunological rejection of heart transplants [35], and ventricular fibrillation in human subjects [36], to mention only a few. Assume that when a biomedical system operates in normal and chaotic mode, one can store the data from the system in a file (measuring a suitable variable at discrete times). Later when the system starts to operate periodically, one can use these data and the above method or a similar coupling to (re)establish chaotic dynamics again in the system (anticontrol of chaos). This example shows that the described synchronization method can also be used to control spatiotemporal chaos.

To conclude, the synchronization method proposed in this Letter can be applied to a pair of unidirectional coupled arrays where the synchronization mechanism of the local elements or cells is known. Synchronization of sporadically coupled individual cells can be achieved, for example, using the subsystem decomposition introduced by Pecora and Carroll [7] or using an active-passive decomposition [14]. Therefore this synchronization strategy for (chaotic) arrays is very general and can be applied to various pairs of coupled systems including two- or three-dimensional arrays, nonhomogeneous arrays, and arrays with different internal couplings. This robustness with respect to details of the implementation makes the approach also useful for practical applications, for example in communication systems where it is easy to build an array of electrical circuits as a monolithic chip.

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[28] For example, for $T = 0.01$ the critical value of the coupling constant is $D_c \approx 2.8$, and for $D = 6$ the critical value of the characteristic time is $T_c \approx 0.0102$. Until now no analytical results are known for the relationship between T_c and D_c. We will present further numerical investigations elsewhere. Usually D_c is smaller than the value for which the cells *within* an array are synchronized.
[29] Of course, the critical values for the coupling constant and the characteristic time depend not only on the vector field of the single cell, but also on the type of internal coupling between cells within the array. In general, the couplings that can synchronize the cells in the array (as diffusive or reciprocal inhibitor couplings [30]) have a tendency to increase the critical values of the parameters.
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