

Classical Decay of the Inflaton

S. Yu. Khlebnikov¹ and I. I. Tkachev^{2,3}

¹*Department of Physics, Purdue University, West Lafayette, Indiana 47907*

²*Department of Physics, The Ohio State University, Columbus, Ohio 43210*

³*Institute for Nuclear Research of the Academy of Sciences of Russia, Moscow 117312, Russia*

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We present the first fully nonlinear calculation of inflaton decay. We map inflaton decay onto an equivalent classical problem and solve the latter numerically. In the $\lambda\phi^4$ model, we find that parametric resonance develops slower and ends at smaller values of fluctuating fields, as compared to estimates existing in literature. We also observe a number of qualitatively new phenomena, including a stage of semiclassical thermalization, during which the decay of inflaton is essentially as effective as during the resonance stage. [S0031-9007(96)00641-2]

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Amplification of quantum fluctuations and transition to semiclassical behavior are familiar subjects in inflationary cosmology (for a review of the latter, see Ref. [1]). Semiclassical fluctuations produced during inflation can explain the observed large scale structure of the Universe [2]. The theory of their production is by now well developed; for a recent rather general presentation, see Ref. [3]. After inflation ends, the scalar field (inflaton) that was driving it begins to oscillate. These oscillations are thought to lead to particle production, inflaton decay, and eventually to reheating of the Universe.

It was only recently realized that phenomena related to large occupation numbers of Bose particles produced by inflaton decay, such as parametric resonance, can be significant in the reheating process [4–6]. That means that semiclassical phenomena may play as important a role in reheating as they do in the structure formation. Of course, the eventual thermalized state will include modes with small occupation numbers, which are essentially quantal, but initial stages of the reheating process should admit a semiclassical description.

However, semiclassical description of fluctuations produced by inflaton decay has not been obtained. The existing treatments of parametric amplification [4–6] employ the standard methods of analyzing particle production, based on Bogoliubov transformation. These methods allow one to take into account *some* of the backreaction effects of produced particles on the oscillating zero-momentum ($\mathbf{k} = 0$) mode, such as time dependence of the frequency of oscillations, but cannot reliably take into account many other nonlinear effects that become important certainly no later than the frequency change. These effects include scattering of produced particles off the $\mathbf{k} = 0$ mode, which knocks particles out of the $\mathbf{k} = 0$ state; rescattering of the produced particles, which populates nonresonant modes as well as returns particles back to $\mathbf{k} = 0$ (Bose condensation) [7], etc. We generically refer to these effects as *rescattering*; we will see that they begin as essentially classical interactions.

Order of magnitude estimates of the amplitude squared of produced fluctuations $(\delta\phi)^2$ at the end of the resonance stage were obtained in Refs. [4–6]. It is important to know $(\delta\phi)^2$ more precisely, because in realistic inflationary models, its magnitude determines whether any symmetries are restored in the nonthermal regime after the resonance stage [8]. This, in turn, requires taking into account all rescattering effects. After the parametric resonance ends, rescattering leads to chaotic behavior and begins to thermalize the system, while it is still in the semiclassical regime. This semiclassical thermalization, which should proceed rather fast due to the still large occupation numbers, is very important for determining the reheating temperature. However, it is completely absent from the standard treatment [4,5], as well as from the one augmented by time-dependent Hartree approximation [6] (in the Hartree approximation, chaotic behavior characteristic of a nonlinear classical system is lost).

It is clear, on the other hand, that if one succeeds in finding an adequate semiclassical description of fluctuations produced by inflaton decay the decay can be studied via the *classical* equations of motion and thus treated as a fully nonlinear problem, without any further assumptions. That classical problem can be turned over to a computer. To obtain the semiclassical description, one has to find the initial conditions for the classical equations of motion, corresponding to the original quantum state. We find these initial conditions using the following observation. Because inflationary models have small coupling constants, fluctuating modes can grow to large occupation numbers before their interactions become important. In other words, the transition to classical behavior occurs in a linear regime with respect to fluctuations. We can then use the linear theory of quantum-to-classical transitions [3] to obtain the initial conditions for subsequent nonlinear classical evolution. Possibility of a semiclassical formulation of the problem of inflaton decay was noted in a recent paper by Son [9]; however, the correct initial conditions for the classical problem were not obtained there.

In this Letter we report the first results of our numerical study of the semiclassical decay of the inflaton, for the simplest model of a real scalar field with the $\lambda\phi^4/4$ potential. Initial conditions were chosen to correspond to conformal vacuum for fluctuations at the end of inflation [10]. Numbers for various quantities of physical interest are presented below; because ours is the first fully nonlinear treatment of the problem, they differ from estimates existing in the literature. We also observe a number of qualitatively new phenomena, including the stage of inflaton decay when the resonance peaks in power spectrum begin to interact and smear out; we call this stage semiclassical thermalization. We find that, during semiclassical thermalization, inflaton decay is essentially as effective as during the stage of parametric resonance.

We consider a real scalar field minimally coupled to gravity, in a Friedmann-Robertson-Walker universe. We work with rescaled variables: time τ , such that $a(\tau)d\tau = \sqrt{\lambda}\phi_0(0)a(0)dt$, coordinate $\xi = \sqrt{\lambda}\phi_0(0)a(0)\mathbf{x}$, and field $\varphi = \phi a(\tau)/\phi_0(0)a(0)$, where $a(\tau)$ is the scale factor, and $\phi_0(0)$ is the zero-momentum mode at $t = \tau = 0$. In this Letter we consider only the massless inflaton case. (The method itself applies to other models, as well as to various laboratory systems.) The action for the scalar field is

$$S = \frac{1}{\lambda} \int d^3\xi d\tau \left[\frac{1}{2} \left(\dot{\varphi} - \frac{\dot{a}}{a} \varphi \right)^2 - \frac{(\nabla_{\xi} \varphi)^2}{2} - \frac{\varphi^4}{4} \right], \quad (1)$$

where dots denote derivatives with respect to τ . The parameter $\dot{a}/a \equiv h$ is the rescaled Hubble parameter, $h(\tau) = H(\tau)/\sqrt{\lambda}\phi_0(0)$. For the massless inflaton, $H(0) \sim \sqrt{\lambda}\phi_0^2(0)/M_{\text{Pl}}$, so that $h(0)$ is independent of λ and is of order 1. The equation of motion following from (1) is

$$\ddot{\varphi} - \nabla_{\xi}^2 \varphi - \frac{\ddot{a}}{a} \varphi + \varphi^3 = 0. \quad (2)$$

Although the coupling λ does not appear in (2), it appears in the action and hence regulates the magnitude of initial quantum fluctuations in nonzero modes relative to the magnitude of the zero mode. We have checked numerically that in the massless case the condition for a semiclassical regime to occur is precisely the smallness of the coupling λ . The zero mode is semiclassical from the beginning. Its initial value is $\varphi_0(0) = 1$, by virtue of our rescaling. We choose the first time when $\dot{\varphi}_0 = 0$ as the definition of moment $\tau = 0$ when the oscillations of inflaton start. We have verified that after that moment the Universe in our model is radiation dominated to very good accuracy, so we can set $\ddot{a} = 0$ in Eq. (2). We make no further assumptions about the zero mode: Its time dependence will come out of the solution to the full nonlinear problem, once we specify the initial conditions for nonzero modes.

According to the preceding discussion, to obtain the initial data for the classical problem, we can linearize Eq. (2) with respect to fluctuations. The linearized equation of motion for the Fourier transform of the fluctuation field (comoving momentum $\mathbf{k} \neq 0$) is

$$\ddot{\varphi}_{\mathbf{k}} + \omega_{\mathbf{k}}^2(\tau)\varphi_{\mathbf{k}} = 0, \quad (3)$$

where $\omega_{\mathbf{k}}^2(\tau) = 3\varphi_0^2(\tau) + k^2$. The operator solution to Eq. (3) is $\varphi_{\mathbf{k}}(\tau) = f_{\mathbf{k}}(\tau)b_{\mathbf{k}}(0) + f_{\mathbf{k}}^*(\tau)b_{-\mathbf{k}}^{\dagger}(0)$, where c -number function $f_{\mathbf{k}}(\tau)$ satisfies Eq. (3) and the initial conditions

$$f_{\mathbf{k}}(0) = \left(\frac{\lambda}{2\omega_{\mathbf{k}}(0)} \right)^{1/2}, \quad \dot{f}_{\mathbf{k}}(0) = [-i\omega_{\mathbf{k}}(0) + h(0)]f_{\mathbf{k}}(0). \quad (4)$$

Operators b^{\dagger} and b are the creation and annihilation operators defined at zero time and normalized according to $[b_{\mathbf{k}}, b_{\mathbf{k}'}^{\dagger}] = \delta(\mathbf{k} - \mathbf{k}')$. A mode becomes semiclassical if the corresponding function $f_{\mathbf{k}}(\tau)$ grows (exponentially) for large τ (and so does, then, the corresponding occupation number). In that case, it can be shown [3] that up to exponentially small corrections, $f_{\mathbf{k}}$ at large τ can be made real by a time-independent phase rotation. When the exponentially small terms are neglected, $\varphi_{\mathbf{k}}(\tau)$ and its canonical momentum commute, and therefore $\varphi_{\mathbf{k}}(\tau)$ at large times can be regarded as a random classical variable. This is the semiclassical description.

The distribution of values of $\varphi_{\mathbf{k}}(\tau)$ is obtained from the solution to the linearized quantum problem. It depends on the quantum state for fluctuations existing at the end of inflation. Below we present results obtained using conformal vacuum as the original quantum state, which should be a good approximation for all $k \geq 1$ (subhorizon modes) [10]. The corresponding classical distribution function at large times is, cf. Ref. [3],

$$\mathcal{F}[\varphi, \dot{\varphi}; \tau] = \mathcal{N} \exp\left(-\int' \frac{|\varphi_{\mathbf{k}}|^2}{|f_{\mathbf{k}}(\tau)|^2} d^3k\right) \times \delta(f_{\mathbf{k}}(\tau)\dot{\varphi}_{\mathbf{k}} - \dot{f}_{\mathbf{k}}(\tau)\varphi_{\mathbf{k}}), \quad (5)$$

where the delta function is a shorthand for the product of functional delta functions for real and imaginary parts, and the prime on the integral reminds us that the integral is taken over half of the values of \mathbf{k} (since these give all independent $\varphi_{\mathbf{k}}$) and does not include $\mathbf{k} = 0$; \mathcal{N} is a time-independent normalization. A useful point here is that (5) satisfies the classical Liouville equation during the entire linear stage. So, we can evolve it back and use it as an initial condition at $\tau = 0$, where $f_{\mathbf{k}}$ and its derivative are given by Eq. (4).

We thus use a discretized version of the distribution

$$\mathcal{P}[\varphi(0)] = N \exp\left(-\frac{2}{\lambda} \int' \omega_{\mathbf{k}}(0)|\varphi_{\mathbf{k}}(0)|^2 d^3k\right) \quad (6)$$

to generate random initial values of $\varphi_{\mathbf{k}}$. Once a value of $\varphi_{\mathbf{k}}$ is generated, the corresponding initial ‘‘velocity’’

is determined uniquely, according to Eqs. (4) and (5), as $\dot{\varphi}_{\mathbf{k}}(0) = [-i\omega_{\mathbf{k}}(0) + h(0)]\varphi_{\mathbf{k}}(0)$. Together with the initial conditions for the zero mode, this forms (after an inverse Fourier transform) an initial data problem for the fully nonlinear equation (2).

We have integrated numerically Eq. (2) in a box of finite size L with the above initial conditions. We have varied the size of the box and the number of grid points N in a box of a given size to make sure that we are close to the continuum limit. We have varied the coupling constant λ over the range 10^{-10} –1. Here we present the results for $L = 16\pi$, $N = 128^3$, and $\lambda = 10^{-4}$. All quantities we measured are averages over many coordinate or momentum points; for these quantities, there is no need in further averaging over initial conditions.

One important quantity that can be measured is the power spectrum of fluctuations, $P(k)$. It is proportional to $\varphi_{\mathbf{k}}^* \varphi_{\mathbf{k}}$ averaged over the direction of \mathbf{k} and is normalized in such a way that Parseval's theorem reads $\int d^3k P(k) = L^{-3} \int d^3\xi [\varphi(\xi) - \varphi_0]^2$. Thus normalized power spectrum does not depend on the size of the (large) box. It is presented in Fig. 1 for several moments of time. Resonance peaks develop at $\tau < 200$ in the following order: first at $k \approx 1$, somewhat later at k close to zero, and still later at larger momenta. For comparison, we have run the problem linearized with respect to fluctuations. Only the initial development of the first peak was unaffected by linearization. The peak at $k \approx 2$ was barely visible, the others were not seen at all. We conclude that all peaks in Fig. 1, except the one at $k \approx 1$, are due to rescattering. The peak at k close to zero is not expected from the approximate linear analysis based on the Mathieu equation. We interpret it as a result of the rescattering process in which a particle from the first peak transfers some of its momentum to a particle from the $\mathbf{k} = 0$ condensate [11]. The positions of the $k \approx 2$ and higher peaks are close to those predicted by the Mathieu equation, but their widths and magnitudes are not. By suppressing the initial fluctuations for $k > 1.75$, we have verified that these peaks are due mainly to rescattering processes $1 + 1 \rightarrow 0 + 2$, $1 + 2 \rightarrow 0 + 3$, etc., where 0 denotes a particle from the

condensate, and 1, 2, 3 particles form the first, second, and third peaks, respectively. Thus, the rescattering processes already start to play some role at the parametric resonance stage.

Another interesting quantity is the integral of the power spectrum of produced fluctuations over all (nonzero) k . This is simply the variance $\text{Var}(\varphi) = \langle [\varphi(x) - \langle \varphi \rangle]^2 \rangle$ and can be measured independently. The brackets denote averaging over the spatial lattice, so $\langle \varphi \rangle$ is just the zero-momentum mode. The lattice effectively subtracts contribution of high-momentum modes, so at small λ our $\text{Var}(\varphi)$ approximates an already subtracted continuum quantity, equal to the continuum $\text{Var}(\varphi)$ minus its value at zero time. The behavior of $\text{Var}(\varphi)$ with time is shown in Fig. 2. Initially it grows exponentially; this is the stage of parametric resonance. Using logarithmic plots, we have found the following analytical fit for this stage:

$$\ln \text{Var}(\varphi) = 2\mu\tau + \ln\lambda - 8. \quad (7)$$

Because $\text{Var}(\varphi)$ oscillates, the choice of constant here is somewhat arbitrary; we chose it in such a way that the line (7) passes through the maximum of the amplitude of $\text{Var}(\varphi)$ at the end of the resonance stage (located at $\tau \equiv \tau_* \approx 200$ if $\lambda = 10^{-4}$). For the rate of the exponential growth 2μ , we obtain $2\mu = 0.07$ for all small enough λ . This value is smaller than the one obtained from the Mathieu equation arising when the time dependence of the zero mode (an elliptic function in this regime) is replaced by sine with the same period. When we make this replacement in our program, we obtain for the rate the value of 0.11, in our time units, as expected. As another test, we have varied the size of the box and verified that we do not miss the fastest growing mode.

The value of $\text{Var}(\varphi)$ in the maximum at the end of the resonance stage is approximately 0.07 and does not depend on λ when $\lambda \ll 1$. Thus, for general (small) λ the resonance ends at time $\tau_* = 76 - 14.3 \ln\lambda$. Extrapolating to a realistic value $\lambda = 10^{-13}$, we find $\tau_* \approx 500$. To obtain the variance of the physical field ϕ at time τ , we need, at large τ , to multiply $\text{Var}(\varphi)$ by $3M_{\text{Pl}}^2/2\pi\tau^2$. Hence, the maximum variance of ϕ , at the end of the

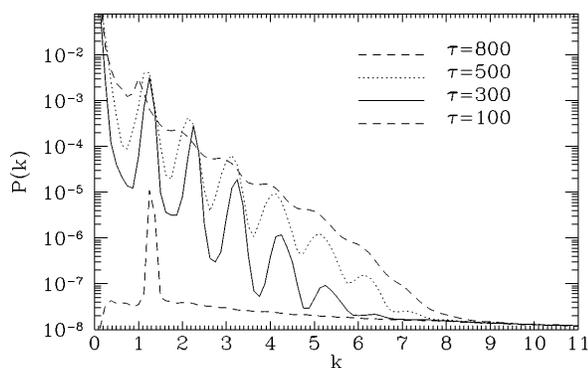


FIG. 1. Power spectrum of fluctuations at successive moments of time.

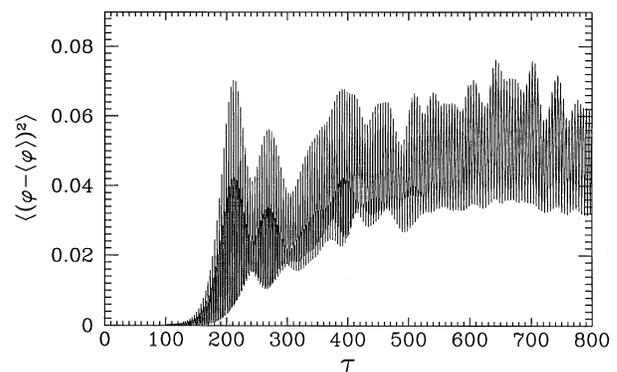


FIG. 2. Variance of the scalar field as a function of time.

resonance stage, is

$$\text{Var}(\phi) = 1 \times 10^{-7} M_{\text{Pl}}^2. \quad (8)$$

For the effective temperature T_{eff} , defined by equating (8) to $T_{\text{eff}}^2/12$, this gives $T_{\text{eff}} = 1 \times 10^{-3} M_{\text{Pl}}$. Notice that T_{eff} does not depend on the initial value of the inflaton field after inflation. The smallness of T_{eff} is a result of two factors not taken into account previously: $\text{Var}(\varphi) \ll 1$ and very large τ_* . In realistic models, including more fields, T_{eff} determines if any symmetries are restored in the nonthermal regime after the resonance stage [8] (Ref. [12] considers nonthermal symmetry restoration for first-order inflation). If, in those models, T_{eff} remains 3 orders of magnitude lower than the Planck scale, then, for example, nonthermal restoration of grand unified theory symmetry may be prevented. On the other hand, we see that T_{eff} is much larger than the reheating temperature that would be obtained if one neglected the effect of large occupation numbers, which confirms the main point of Ref. [8].

Finally, the time dependence of the zero-momentum mode is shown in Fig. 3. We can see that, after decreasing during the parametric resonance, it rebounds twice during time $\tau = 200$ to 300. We attribute this to Bose condensation caused by rescattering. Comparison of Figs. 2 and 3 suggests that it is Bose condensation that terminates parametric resonance. However, the magnitude of the zero mode continues to decrease significantly, approximately as $\tau^{-1/3}$ (as opposed to a slower $\tau^{-1/6} \propto t^{-1/12}$ decay found in Ref. [4]). Returning to Fig. 1, we see that the peaks smear out at this stage. This is the onset of classical chaos or, in the language of particle physics, stimulated scattering, decay, and annihilation due to large occupation numbers. This process is rather effective. In our model the energy scales as the fourth power of the field. So, for $\lambda = 10^{-4}$, at time $\tau \approx 400$ about 70% of energy is in fluctuations. For smaller λ , the onset of this chaotic regime, which we call semiclassical thermalization, is delayed, but its rate with respect to time τ does not decrease. The reason is that the strength of rescattering is determined not by λ itself but by the product of λ and a relevant occupation number.

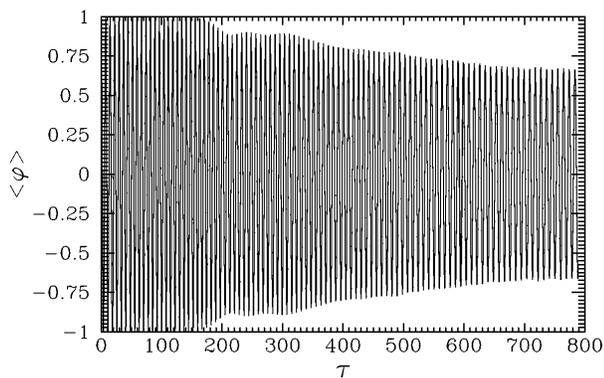


FIG. 3. Time dependence of the zero-momentum mode.

In conclusion, we have done the first fully nonlinear calculation of inflaton decay. We have mapped the inflaton decay onto an equivalent classical problem and solved the classical problem numerically. In the $\lambda\phi^4$ model, we have found that parametric resonance develops slower and ends at smaller values of fluctuating fields, as compared to estimates existing in the literature. We have also observed a number of qualitatively new phenomena, including a stage of semiclassical thermalization (chaos), during which the decay of inflaton is essentially as effective as during the resonance stage.

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- [1] A. D. Linde, *Particle Physics and Inflationary Cosmology* (Harwood, Chur, Switzerland, 1990); E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, Redwood City, California, 1990).
 - [2] V. F. Mukhanov and G. V. Chibisov, *JETP Lett.* **33**, 532 (1981); S. W. Hawking, *Phys. Lett.* **115B**, 295 (1982); A. A. Starobinsky, *Phys. Lett.* **117B**, 175 (1982); A. H. Guth and S.-Y. Pi, *Phys. Rev. Lett.* **49**, 1110 (1982).
 - [3] D. Polarski and A. A. Starobinsky, Report No. gr-qc/9504030 (to be published).
 - [4] L. Kofman, A. Linde, and A. A. Starobinsky, *Phys. Rev. Lett.* **73**, 3195 (1994).
 - [5] J. H. Traschen and R. H. Brandenberger, *Phys. Rev. D* **42**, 2491 (1990); Y. Shtanov, J. Traschen, and R. Brandenberger, *Phys. Rev. D* **51**, 5438 (1995).
 - [6] D. Boyanovsky, H. J. de Vega, R. Holman, D.-S. Lee, and A. Singh, *Phys. Rev. D* **51**, 4419 (1995); see also D. Boyanovsky *et al.*, *Phys. Rev. D* **52**, 6805 (1995).
 - [7] Because Bose condensation is always associated with large occupation numbers, it can be viewed as a classical effect, analogous to classical wave turbulence; see, e.g., B. V. Svistunov, *J. Moscow Phys. Soc.* **1**, 373 (1991); D. V. Semikoz and I. I. Tkachev, *Phys. Rev. Lett.* **74**, 3093 (1995); D. V. Semikoz and I. I. Tkachev, Report No. hep-ph/9507306 (to be published).
 - [8] L. Kofman, A. Linde, and A. A. Starobinsky, *Phys. Rev. Lett.* **76**, 1011 (1996); I. I. Tkachev, Report No. hep-th/9510146 (to be published).
 - [9] D. T. Son, Report No. hep-ph/9601377 (to be published).
 - [10] We tried other distributions as well; the results are rather insensitive to the form of initial distribution, as long as the initial fluctuations are of the correct order of magnitude.
 - [11] This is apparently the strongest rescattering process at this stage, because it takes advantage of both the existence of the condensate and the already large occupation number in the first peak.
 - [12] E. Kolb and A. Riotto, Report No. astro-ph/9602095 (to be published).