Systematics of Soft Final-State Interactions in B Decays

John F. Donoghue, Eugene Golowich, Alexey A. Petrov, and João M. Soares
Department of Physics and Astronomy, University of Massachusetts, Amherst, Massachusetts 01003
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By using very general and well established features of soft strong interactions we show, contrary to conventional expectations, that (i) soft final-state interactions (FSI) do not disappear for large m_B, (ii) inelastic rescattering is expected to be the main source of soft FSI phases, and (iii) FSI which interchange charge and/or flavors are suppressed by a power of m_B, but are quite likely to be significant at $m_B \simeq 5$ GeV. We briefly discuss the influence of these interactions on tests of *CP* violation and on theoretical calculations of weak decays. [S0031-9007(96)01152-0]

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It is notoriously difficult to say anything useful about final-state interactions in weak decays. Although the final-state interactions are not themselves of fundamental interest, they are important for some truly interesting aspects of B decay. For example, many signals of direct *CP* violation in B transitions require final-state phases as well as *CP*-violating phases if the *CP*-odd asymmetry is to be nonzero [1]. In this paper we shall derive some general properties of soft final-state interactions and describe the implications for theory and phenomenology.

The scattering of hadrons at high energies exhibits a two-component structure of "soft" and "hard" scattering. Soft scattering is that which occurs primarily in the forward direction. The transverse momentum is limited, having a distribution which falls exponentially with a scale of order 0.5 GeV. At higher transverse momentum, ultimately one encounters the region of hard scattering, which falls only as a power of the transverse momentum. Collisions involving hard scattering are interpreted as interactions between pointlike constituents of the hadrons, the quarks, and gluons of QCD. These are calculable in QCD perturbation theory and are found to be in good quantitative agreement with experiment. Hard scattering is, however, only a very small portion of the total hadronic cross section. The much larger soft component at low values of transverse momentum is by far the dominant contribution to high energy scattering. Although soft hadronic interactions are generally not calculable from first principles, there is available a wealth of experimental studies [2] and accurate high energy phenomenology [3] on which to base our study.

The modern approach to B physics employs as an organizing principle the fact that the B mass is very large compared to the QCD scale. In the context of soft final-state interactions (FSI) in B decays, it suggests the question—what is the leading order behavior of soft final-state phases in the $m_B \to \infty$ limit? One perception is that they might become less and less important as the mass of the decaying quark becomes heavier. This is because, roughly speaking, the final-state particles emerge at such high momenta that they do not have a chance to rescatter [4]. Such an expectation is, however, false because soft scattering actually grows with energy. As an example of this important energy dependence, we shall demonstrate below that the imaginary part of the forward elastic amplitude has an $s^{1+\eta}$ ($\eta \simeq 0.08$) dependence, and, as a consequence, the elastic final-state interaction is roughly constant as a function of m_B. (The small exponent $\eta \simeq 0.08$ occurs repeatedly throughout this analysis, but is not in itself of basic significance as our conclusions would be qualitatively unchanged with $\eta = 0$.) We shall then use this observation as the starting point for a more general exploration of the systematics of FSI for large m_B. The inevitability of our conclusions will be seen to follow rather directly from well established aspects of strong interaction phenomenology.

Final-state interactions in B decay involve the rescattering of physical final-state particles. Unitarity of the S matrix, $S^\dagger S = 1$, implies that the $\mathcal{T}$ matrix, $S = 1 + i\mathcal{T}$, obeys

$$\begin{aligned}\text{Disc}\,\mathcal{T}_{B\to f} &\equiv \frac{1}{2i}[\langle f|\mathcal{T}|B\rangle - \langle f|\mathcal{T}^\dagger|B\rangle] \\ &= \frac{1}{2}\sum_I \langle f|\mathcal{T}^\dagger|I\rangle\langle I|\mathcal{T}|B\rangle. \qquad (1)\end{aligned}$$

Of interest are all physical intermediate states which can scatter into the final-state f. Among all these, however, we shall first concentrate on just the *elastic* channel and demonstrate that elastic rescattering does not disappear in the limit of large m_B. (We stress that we are *not* suggesting the elastic channel to be the dominant contribution to soft rescattering. Our analysis leads to quite the opposite conclusion, that it is the inelastic channels which are most important.) The elastic channel is especially convenient for our discussion because we can use the optical theorem to rigorously connect it to known physics. The optical theorem relates the forward invariant amplitude $\mathcal{M}$ to the total cross section,

$$\text{Im}\,\mathcal{M}_{f\to f}(s, t = 0) = 2k\sqrt{s}\,\sigma_{f\to \text{all}} \sim s\sigma_{f\to\text{all}}\,, \qquad (2)$$

where s is the squared center-of-mass energy and t is the squared momentum transfer.

The asymptotic total cross sections are known experimentally to rise slowly with energy. All known cross

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sections can be parametrized by fits of the form [5]

$$\sigma(s) = X(s/s_0)^{0.08} + Y(s/s_0)^{-0.56}, \qquad (3)$$

where $s_0 = \mathcal{O}(1)$ GeV is a typical hadronic scale. Thus, the imaginary part of the forward elastic scattering amplitude rises asymptotically as $s^{1.08}$. This growth with s is counterintuitive in that it cannot be generated by a perturbative mechanism at any finite order. In particular, calculations based on the quark model or perturbative QCD would completely miss this feature.

In order to arrive most simply at our goal, let us first consider only this imaginary part, and build in the known exponential falloff of the elastic cross section in t (recalling that t is negative) by writing

$$i \,\mathrm{Im}\, \mathcal{M}_{f\to f}(s,t) \simeq i\beta_0(s/s_0)^{1.08}e^{bt}. \qquad (4)$$

It is then an easy task to calculate the contribution of the imaginary part of the elastic amplitude to the unitarity relation for a final-state $f = a + b$ with kinematics $p_a' + p_b' = p_a + p_b$ and $s = (p_a + p_b)^2$, and we find

$$\begin{aligned}\mathrm{Disc}\,\mathcal{M}_{B\to f} &= \frac{1}{2}\int \frac{d^3p_a'}{(2\pi)^3 2E_a'}\frac{d^3p_b'}{(2\pi)^3 2E_b'}(2\pi)^4\delta^{(4)}(p_B - p_a' - p_b')\left[-i\beta_0\left(\frac{s}{s_0}\right)^{1.08} e^{b(p_a-p_a')^2}\mathcal{M}_{B\to f}\right] \\ &= -\frac{i}{32\pi}\int d(\cos\theta)e^{-(bs/2)(1-\cos\theta)}\beta_0\left(\frac{s}{s_0}\right)^{1.08}\mathcal{M}_{B\to f} = -\frac{1}{16\pi}\frac{i\beta_0}{s_0 b}\left(\frac{m_B^2}{s_0}\right)^{0.08}\mathcal{M}_{B\to f}\,, \end{aligned} \qquad (5)$$

where we have used $t = (p_a - p_a')^2 \simeq -s(1-\cos\theta)/2$ and have taken $s = m_B^2$. The integration over the angle involving the direction of the intermediate state is seen to introduce a suppression factor to the final-state interaction of $s^{-1} = m_B^{-2}$. [Note that for a final state like $\pi\pi$ which occurs in a definite angular momentum eigenstate (S wave), the final angular integral is the same as projecting out the S wave scattering state. A different way of understanding the kinematic factor of $1/m_B^2$ is that the S wave component of elastic scattering is a fraction $1/bm_B^2$ of the total amplitude.] This is because the soft final-state rescattering can take place only if the intermediate state has a transverse momentum $p_\perp \lesssim 1$ GeV with respect to the final particle direction. This would naively suggest a result consistent with conventional expectations, i.e., an FSI which falls as m_B^{-2}. However, the fact that the forward scattering amplitude *grows* with a power of s overcomes this suppression and leads to elastic rescattering which does not disappear at large m_B.

In fact, we can make a more detailed estimate of elastic rescattering because the phenomenology of high energy scattering is well accounted for by Regge theory [6]. Scattering amplitudes are described by the exchanges of Regge trajectories (families of particles of differing spin) which lead to elastic amplitudes of the form

$$\mathcal{M}_{f\to f} = \xi\beta(t)\,(s/s_0)^{\alpha(t)}e^{i\pi\alpha(t)/2}, \qquad (6)$$

with $\xi = 1$ for charge conjugation $C = +1$ and $\xi = i$ for $C = -1$. Each such trajectory is described by a straight line,

$$\alpha(t) = \alpha_0 + \alpha' t\,. \qquad (7)$$

The leading trajectory for high energy scattering is the Pomeron, having $C = +1, \alpha_0 \simeq 1.08$, and $\alpha' \simeq 0.25$ GeV^{-2}. Note that since

$$(s/s_0)^{\alpha(t)} = (s/s_0)^{\alpha_0}e^{\alpha'\ln[s/s_0]t}, \qquad (8)$$

the exponential falloff in t is connected with the slope α' and the effective slope parameter b in Eq. (4) thus increases logarithmically with s. Since α_0 is near unity, the phase of the Pomeron-exchange amplitude is seen from Eq. (6) to be almost purely imaginary. This feature has been verified experimentally by interference measurements. There are several next-to-leading trajectories, both those with $C = -1$ [$\rho(770)$ and $\omega(782)$ trajectories] and those with $C = +1$ [$a_2(1320)$ and $f_2(1270)$ trajectories]. Roughly, these have $\alpha_0 \simeq 0.44$, $\alpha' \simeq 0.94$ GeV^{-2} and lead collectively to the $s^{-0.56}$ dependence in the asymptotic cross section of Eq. (3). The prefactor $\beta(t)$ in Eq. (6) also has known regularities. For the Pomeron, β is very nearly proportional to the number of quarks at each vertex, and carries a power law behavior similar to the electromagnetic form factor. Therefore, $\beta_{\pi\pi}$ in pion-pion scattering can be expressed in terms of the analogous proton-proton quantity β_{pp} as

$$\beta_{\pi\pi}(t) = (\tfrac{2}{3})^2\beta_{pp}(t=0)/(1 - t/m_\rho^2)^2. \qquad (9)$$

The combination of exponential and power law t dependence in a generic Regge amplitude gives a unitarity integral no longer having an elementary form. However, the integration can still be carried out in terms of Euler functions. Taking $s = m_B^2 \simeq 25$ GeV ^{2}, we obtain for the Pomeron contribution

$$\mathrm{Disc}\,\mathcal{M}_{B\to\pi\pi}|_{\mathrm{Pomeron}} = -i\epsilon\mathcal{M}_{B\to\pi\pi}\,, \qquad (10)$$

where we find from our computation,

$$\epsilon \simeq 0.21\,. \qquad (11)$$

From this numerical result and from the nature of its derivation, we may anticipate that additional individual soft FSI will not be vanishingly small. Moreover, other final states should have elastic-rescattering effects of comparable size. However, of chief significance is the weak dependence of ϵ on m_B that we have found—the $(m_B^2)^{0.08}$ factor in the numerator is attenuated by the $\ln(m_B^2/s_0)$ dependence in the effective value of b [compare Eqs. (4) and (8)].

The above study of the elastic channel, although instructive, is far from the whole story. In fact, it suggests the even more significant result that at high energies *FSI phases are generated chiefly by inelastic effects.* At a physical level, this conclusion is forced on us by the fact that the high energy cross section is mostly inelastic. It is also plausible at the analytic level, given that the Pomeron elastic amplitude is almost purely imaginary. The point is simply this. Our study of elastic rescattering has yielded a $\mathcal{T}$-matrix element $\mathcal{T}_{ab\to ab} = 2i\epsilon$, which directly gives $S_{ab\to ab} = 1 - 2\epsilon$. However, the unitarity of the S matrix can be shown to imply that the off-diagonal elements must be $\mathcal{O}(\sqrt{\epsilon})$. Since ϵ is approximately $\mathcal{O}(m_B^0)$ in powers of m_B and numerically $\epsilon < 1$, the inelastic amplitude must also be $\mathcal{O}(m_B^0)$ and of magnitude $\sqrt{\epsilon} > \epsilon$. There is an alternate argument, utilizing the form of the final-state unitarity relations, which also shows that inelastic effects are required to be present. In the limit of T invariance for the weak interactions, the discontinuity Disc $\mathcal{M}_{B\to f}$ is a real number (up to irrelevant rephasing invariance of the B state). The factor of i obtained in the elastic rescattering in Eq. (10) must be compensated for by the inelastic rescattering (this effect is made explicit in the example to follow) in order to make the total real. Therefore, the presence of inelastic effects is seen to be necessary.

Analysis of the final-state unitarity relations in their most general form,

$$\text{Disc}\,\mathcal{M}_{B\to f_1} = \frac{1}{2}\sum_k \mathcal{M}_{B\to k} T^\dagger_{k\to f_1}\,, \tag{12}$$

is quite complicated due to the many contributing intermediate states present at the B mass. However, it is possible to illustrate the systematics of inelastic scattering by means of a simple two-channel model. This pedagogic example involves a two-body final-state f_1 undergoing elastic scattering and a final-state f_2 which is meant to represent "everything else." We assume that the elastic amplitude is purely imaginary. Thus, the scattering can be described in the one-parameter form

$$S = \begin{pmatrix} \cos 2\theta & i\sin 2\theta \\ i\sin 2\theta & \cos 2\theta \end{pmatrix}, \quad T = \begin{pmatrix} 2i\sin^2\theta & \sin 2\theta \\ \sin 2\theta & 2i\sin^2\theta \end{pmatrix}, \tag{13}$$

where, from our elastic-rescattering calculation, we identify $\sin^2\theta \equiv \epsilon$. The unitarity relations become

$$\begin{aligned} \text{Disc}\,\mathcal{M}_{B\to f_1} &= -i\sin^2\theta\,\mathcal{M}_{B\to f_1} + \tfrac{1}{2}\sin 2\theta\,\mathcal{M}_{B\to f_2}\,, \\ \text{Disc}\,\mathcal{M}_{B\to f_2} &= \tfrac{1}{2}\sin 2\theta\,\mathcal{M}_{B\to f_1} - i\sin^2\theta\,\mathcal{M}_{B\to f_2}\,. \end{aligned} \tag{14}$$

Let us denote the real numbers $\mathcal{M}_1^0$ and $\mathcal{M}_2^0$ to be the decay amplitudes in the limit $\theta \to 0$. Then an exact solution to these equations is given by

$$\begin{aligned} \mathcal{M}_{B\to f_1} &= \cos\theta\,\mathcal{M}_1^0 + i\sin\theta\,\mathcal{M}_2^0\,, \\ \mathcal{M}_{B\to f_2} &= \cos\theta\,\mathcal{M}_2^0 + i\sin\theta\,\mathcal{M}_1^0\,. \end{aligned} \tag{15}$$

As a check, we can insert these solutions back into Eq. (14). Upon doing so and bracketing contributions from $\mathcal{M}_{B\to f_1}$ and $\mathcal{M}_{B\to f_2}$ separately, we find

$$\begin{aligned} \text{Disc}\,\mathcal{M}_{B\to f_1} = \tfrac{1}{2}\{&[-2i\epsilon\,\mathcal{M}^0_{B\to f_1} + \mathcal{O}(\epsilon^{3/2})] \\ &+ (2\sqrt{\epsilon}\,\mathcal{M}^0_{B\to f_2} + 2i\epsilon\,\mathcal{M}^0_{B\to f_1})\}\,. \end{aligned} \tag{16}$$

The first of the four terms comes from the elastic channel f_1 and is seen to be canceled by the final term, which arises from the inelastic channel f_2. The third term is dominant, being $\mathcal{O}(\sqrt{\epsilon})$, and comes from the inelastic channel.

In this example, we have seen that the phase is given by the inelastic scattering with a result of order

$$\text{Im}\,\mathcal{M}_{B\to f}/\text{Re}\,\mathcal{M}_{B\to f} \sim \sqrt{\epsilon}\,\mathcal{M}_2^0/\mathcal{M}_1^0\,. \tag{17}$$

Clearly, for physical B decay, we no longer have a simple one-parameter S matrix. However, the main feature of the above result is expected to remain—that inelastic channels cannot vanish because they are required to make the discontinuity real and that the phase is systematically of order $\sqrt{\epsilon}$ from these channels. Of course, with many channels, cancellations or enhancements are possible for the sum of many contributions. In any given process, our analysis cannot rule out the possibility of cancellations suppressing the phase (this would correspond to a small $\mathcal{M}_2^0$ in the simplified analysis above). However, since we have shown that both elastic and inelastic rescattering occurs at order m_B^0, and these conclusions survive the generalization to multiple channels, we must expect that the final-state phases share this same property, aside from possible exceptional cases. In this situation, the generic expectation remains—that inelastic soft final-state rescattering arising from Pomeron exchange will generate a phase which does not vanish in the large m_B limit.

What about nonleading effects? Because the nonleading trajectories involve particles with charge, spin, and net flavor quantum numbers, these exchanges differ from the Pomeron in being able to change these quantum numbers in the rescattering process. It is not hard to see that these may be significant at the physical values of m_B. For example, the fit to the $\bar{p}p$ total cross section is

$$\sigma(p\bar{p}) = \left[22.7\left(\frac{s}{s_0}\right)^{0.08} + 140\left(\frac{s}{s_0}\right)^{-0.56}\right](\text{mb})\,, \tag{18}$$

with $s_0 = 1\ \text{GeV}^2$. At $s = (5.2\ \text{GeV})^2$, the nonleading coefficient is a factor of 6 larger than the leading effect, effectively compensating for the $s^{-0.56} = m_B^{-1.12}$ suppression. The subleading terms are then comparable in the elastic forward $\bar{p}p$ scattering amplitude. The slope of the ρ trajectory and hence the experimental falloff with t is larger than that of the Pomeron by a factor of nearly 4, and thus this moderates the integrated rescattering effects. If we estimate the β coefficient of the ρ trajectory in $\pi\pi$ by relating it to $\bar{p}p$ via a factor of $\beta_{\pi\pi} \simeq 4\beta_{\bar{p}p}$ and then perform the integration over the intermediate state momentum, we find

$$\text{Disc}\,\mathcal{M}_{B\to\pi\pi}|_{\rho-\text{traj}} = i\epsilon_\rho\,\mathcal{M}_{B\to\pi\pi}\,, \tag{19}$$

with $\epsilon_\rho \simeq 0.11 - 0.05i$. It is likely that the $f_2(1270)$ trajectory could be somewhat larger, as it is in $\bar{p}p$ and πp scattering.

Final-state phases can contribute to weak decay phenomenology in a variety of ways. Here, we briefly consider two of these, isospin sum rules and CP-violating asymmetries. A simple example of an isospin sum rule is the following relation between $B \to D\pi$ decay amplitudes,

$$\mathcal{M}_{+0} = \mathcal{M}_{+-} + \sqrt{2}\,\mathcal{M}_{00}, \qquad (20)$$

where $\mathcal{M}_{+0} \equiv \mathcal{M}(B^+ \to \pi^+ \bar{D}^0)$, etc. Measurement of the magnitude of each amplitude via the partial decay rate allows one to observe the relative final-state phases of the different isospin components. Noting that the $D\pi$ final state in B decay occurs in the isospin states $I = 1/2, 3/2$, one can solve for the difference in phase angles,

$$\cos(\delta_{1/2} - \delta_{3/2}) = \sqrt{\frac{1}{8}}\,\frac{3|\mathcal{M}_{+-}|^2 - 6|\mathcal{M}_{00}|^2 + |\mathcal{M}_{+0}|^2}{|\mathcal{M}_{+0}|\sqrt{3|\mathcal{M}_{+-}|^2 + 3|\mathcal{M}_{00}|^2 - |\mathcal{M}_{+0}|^2}}. \qquad (21)$$

The key qualitative property of Pomeron exchange is that it is independent of the charge states of the external particles, being identical for each $D^i\pi^j$ final state. However, the relative sizes of the weak amplitudes can be different in the different channels. Since [cf. Eq. (12)] the final-state phase involves a product of strong and weak amplitudes, it is possible (but not required) that a nonzero phase difference is generated.

CP-violating asymmetries involve comparisons of $B \to f$ and $\bar{B} \to \bar{f}$. In order to be nonzero, these require two different pathways to reach the final state f, and these two paths must involve different *CP*-violating weak phases and different strong phases. The leading Pomeron phases *can* contribute to such asymmetries if the other conditions are met. Because the strong phase is generated by inelastic channels, the relevant pathways would involve $B \to f$ directly or $B \to$ "multibody" followed by the inelastic rescattering, multibody $\to f$. Depending on the dynamics of weak decay matrix elements, these may pick up different weak phases. As an example, consider the final state $f = K^-\pi^0$, which can be generated either by a standard W exchange or by the penguin diagram, involving different weak phases [7]. For the strong rescattering, we must also consider a channel to which $K^-\pi^0$ scatter inelastically, which we call $Kn\pi$ (although one can generate this asymmetry by a hard rescattering $D_sD \to K^-\pi^0$, we are concentrating here on the soft physics). The W-exchange and penguin amplitudes will contribute with different weights to $K\pi$ and $Kn\pi$, so that in the absence of final-state interactions we expect

$$\begin{aligned}\mathcal{M}(B^- \to K^-\pi^0) &= |A_1|e^{i\phi_1} = A_1^w e^{i\phi_w} + A_1^p e^{i\phi_p},\\ \mathcal{M}(B^- \to Kn\pi) &= |A_n|e^{i\phi_n} = A_n^w e^{i\phi_w} + A_n^p e^{i\phi_p},\end{aligned} \qquad (22)$$

with $\phi_1 \neq \phi_n$. If we now model the strong rescattering by the two-channel model described above, we have for B and $\bar{B}$ decays

$$\begin{aligned}\mathcal{M}(B^- \to K^-\pi) &= |A_1|e^{i\phi_1} + i\sqrt{\epsilon}\,|A_n|e^{i\phi_n},\\ \mathcal{M}(B^+ \to K^+\pi) &= |A_1|e^{-i\phi_1} + i\sqrt{\epsilon}\,|A_n|e^{-i\phi_n}.\end{aligned} \qquad (23)$$

This leads to a *CP*-violating decay rate asymmetry

$$\Gamma(B^- \to K^-\pi^0) - \Gamma(B^+ \to K^+\pi^0) \sim \sqrt{\epsilon}\,|A_1||A_n|\sin(\phi_n - \phi_1). \qquad (24)$$

While this effect will be very difficult to calculate, we see that inelastic final-state interactions can contribute to *CP*-violating asymmetries at leading order in m_B.

The results obtained in this paper must also be accounted for in any theoretical calculation of weak decay amplitudes. For large m_B, there is the hope that one can directly calculate the weak matrix elements through variants of the factorization hypothesis or by perturbative QCD. Final-state interactions will impose limits on the accuracy of such methods, as no existing technique includes the effect of inelastic scattering. There must exist, in every valid theoretical calculation, a region of the parameter space where the nonperturbative Regge physics is manifest. Arguments based on local quark-hadron duality do not account for these effects of soft physics because the growth of the scattering amplitude with s (for both the leading and first nonleading trajectories) cannot be seen in perturbative calculations. It remains an intriguing possibility that the assumption of quark-hadron duality can be questioned in other aspects of B decay as well. At any rate, for final-state interaction studies, one may only hope that the perturbative and calculable physics is larger than the difficult nonperturbative contributions discussed in this paper.

To conclude, we have argued that the general features of soft scattering have forced upon us some surprising conclusions regarding final-state interactions. Most importantly, the growth of forward scattering with s, as required by the optical theorem and cross section data, indicates that soft scattering does not decrease for large m_B (the dependence would be weak even if total cross sections were constant in the energy). The structure of the elastic rescattering via the Pomeron also indicates that *inelastic* processes are expected to be the leading sources of strong phases. These systematics can be important for the phenomenology of B decays.

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