

Exact Solution of the One-Dimensional Non-Abelian Coulomb Gas at Large N

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The problem of computing the thermodynamic properties of a one-dimensional gas of particles which transform in the adjoint representation of the gauge group and interact through non-Abelian electric fields is formulated and solved in the large N limit. The explicit solution exhibits a first order confinement-deconfinement phase transition with computable properties and describes two-dimensional adjoint QCD in the limit where matter field masses are large. [S0031-9007(96)00978-7]

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Two-dimensional quantum chromodynamics (QCD) with adjoint representation matter fields is the simplest field theoretical model which exhibits some of the important common features of string theory and the confining phases of gauge theory. Most notable are the infinite number of asymptotically linear Regge trajectories and a density of states which increases exponentially with energy. In two dimensions, the Yang-Mills field itself has no propagating degrees of freedom. In adjoint QCD, the matter fields provide dynamics by playing a role analogous to the transverse gluons of higher dimensional gauge theory. In fact, dimensional reduction of three-dimensional Yang-Mills theory produces two-dimensional QCD with massless adjoint scalar quarks. Moreover, since adjoint matter fields do not decouple in the infinite N limit, the large N expansion is of a similar level of complexity to that of higher dimensional Yang-Mills theory. One would expect it to exhibit some of the stringy features of the confining phase which are emphasized in that limit.

Although adjoint QCD is not explicitly solvable, even at infinite N , details of its spectrum were readily analyzed by approximate and numerical techniques [1–4]. In addition, Kutasov [2] exploited an argument which was originally due to Polchinski [5] to show that the confining phase must be unstable at high temperature and suggested it as a tractable model where the confinement-deconfinement transition could be investigated.

In this Letter, we shall formulate and find an explicit solution of the large N limit of a simplified version of adjoint QCD. We shall consider a one-dimensional gas of nondynamical particles which have adjoint color charges and which interact with each other through non-Abelian electric fields. Because there are no dynamical gluons which could screen adjoint charges in one dimension, at low temperature and density, adjoint quarks are confined. (In higher dimensions, an adjoint charge and a gluon could form a color singlet bound state.) They form

colorless “hadron” bound states with two or more adjoint quarks connected by nondynamical strings of electric flux. The large N limit resembles a noninteracting string theory in that, at infinite N , the energy of a state is proportional to the total length of all strings of electric flux plus a chemical potential times the total number of quarks. The property of confinement is defined by estimating the energy required to introduce an external fundamental representation quark-antiquark pair into the system. In the confining phase, where the hadron gas is dilute, the quark-antiquark energy is proportional to the length of the electric flux string which, to obtain gauge invariance, must connect them. This gives them a confining interaction. Some typical confined configurations are depicted in Fig. 1. In the confined phase, the average particle number density and the energy density are small—in the large N limit both are of order one, rather than N^2 which one would expect from naive counting of the degrees of freedom. This is consistent with the fact that in a confining phase the number of degrees of freedom, i.e., hadrons, is independent of N . In contrast, in the deconfined phase, since the number of degrees of freedom, i.e., quarks and gluons, is proportional to N^2 the particle density and energy are also of order N^2 .

As temperature or density is increased, eventually we arrive at the situation where there are electric flux strings almost everywhere. Then, adding an additional flux string, or modifying the existing network of strings to accommodate a fundamental representation quark-antiquark pair

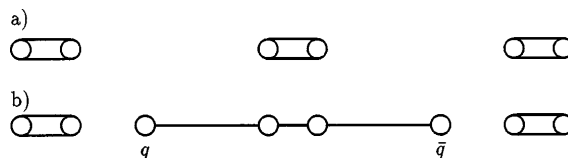


FIG. 1. Some examples of states in the confining phase. Each adjoint quark connects with two electric flux strings and fundamental quarks in (b) connect with one string.

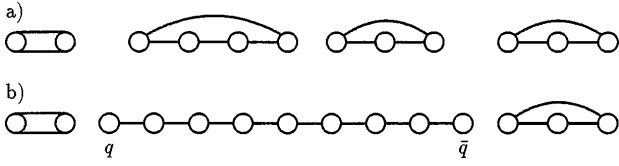


FIG. 2. Typical configurations in the deconfined phase.

involves a negligibly small addition to the energy of the total system (see Fig. 2). This is typical of the deconfined phase.

Between these two phases is a transition, which we show in this Letter, is of first order. In the string picture, this phase transition occurs when the strings in a typical configuration percolate in the one-dimensional space. The order parameter is the Polyakov loop operator [6,7] which measures the exponential of the negative of the free energy which is required to insert a single, unpaired fundamental representation quark source into the system. This free energy is infinite (and the expectation value of the Polyakov loop is zero) in the confining phase and it is finite in the deconfined phase.

In the Hamiltonian formulation of two-dimensional Yang-Mills theory, the electric field is the canonical conjugate of the spatial component of the gauge field, $[A^a(x), E^b(y)] = i\delta^{ab}(x-y)$. The Hamiltonian is

$$H = \int dx \frac{e^2}{2} \sum_{a=1}^{N^2} [E^a(x)]^2, \quad (1)$$

and the Gauss' law constraint, which takes the form of a physical state condition, is

$$\left(\frac{d}{dx} E^a(x) - f^{abc} A^b(x) E^c(x) + \sum_{i=1}^K T_i^a \delta(x-x_i) \right) \times \Psi_{\text{phys}} = 0. \quad (2)$$

[Here, for concreteness, we consider $U(N)$ gauge theory. The gauge field $A = A^a t_a$, with t_a the generators in the fundamental representation.] There are particles with adjoint representation color charges located at positions x_1, \dots, x_K . T_i^a are generators in the adjoint representation operating on the color degrees of freedom of the i th particle.

In the functional Schrödinger picture, the states are functionals of the gauge field and the electric field is the functional derivative operator $E^a(x) = \frac{1}{i} \frac{\delta}{\delta A^a(x)}$. The functional Schrödinger equation is that of a free particle

$$\int dx \left(-\frac{e^2}{2} \sum_{a=1}^{N^2} \frac{\delta^2}{[\delta A^a(x)]^2} \right) \Psi^{a_1 \dots a_K}[A; x_1, \dots, x_K] = \mathcal{E} \Psi^{a_1 \dots a_K}[A; x_1, \dots, x_K], \quad (3)$$

Gauss' law implies that the physical states, i.e., those which obey the gauge constraint (2), transform as

$$\Psi^{a_1 \dots a_K}[A^g; x_1, \dots, x_K] = g_{a_1 b_1}^{Ad}(x_1) \dots g_{a_K b_K}^{Ad}(x_K) \times \Psi^{b_1 \dots b_K}[A; x_1, \dots, x_K], \quad (4)$$

where $A^g \equiv gAg^\dagger - ig\nabla g^\dagger$ is the gauge transform of A . For a fixed number of external charges, this model is explicitly solvable. In the following we shall examine its thermodynamic features, where we assume that the particles have Maxwell-Boltzmann statistics.

We find it convenient to work with the grand canonical ensemble. The partition function is constructed by taking the trace of the Gibbs density $e^{-H/T}$ over physical states. This can be implemented by considering eigenstates of $A^a(x)$ (and an appropriate basis for the nondynamical particles) $|A\rangle e_{a_1} \dots e_{a_K}$. Projection onto gauge invariant states involves a projection operator which has the net effect of gauge transforming the state field at one side of the trace, and integrating over all gauge transformations [8]. The resulting partition function is

$$Z[x_i, T] = \int [dA][dg] \langle A | e^{-H/T} | A^g \rangle \times \text{Tr} g^{Ad}(x_1) \dots \text{Tr} g^{Ad}(x_K), \quad (5)$$

where $[dg(x)]$ is the Haar measure on the space of mappings from the line to the group manifold and $[dA]$ is a measure on the convex Euclidean space of gauge field configurations. For $U(N)$, the trace in the adjoint representation is $\text{Tr} g^{Ad}(x) = |\text{Tr} g(x)|^2$ where $g(x)$ is in the fundamental representation. In order to form the grand canonical ensemble, we average over the particle positions by integrating $\int dx_1 \dots \int dx_K$, multiply by the fugacity to the power K , λ^K , divide by the statistics factor $1/K!$ and sum over K . The result is

$$Z[\lambda, T] = \int [dA][dg] e^{-S_{\text{eff}}[A, g]}, \quad (6)$$

where the effective action is

$$e^{-S_{\text{eff}}[A, g]} = \langle A | e^{-H/T} | A^g \rangle \exp\left(\int dx \lambda |\text{Tr} g|^2 \right). \quad (7)$$

The Hamiltonian is the Laplacian on the space of gauge fields. Using the explicit form of the heat kernel $\langle A | e^{-H/T} | A^g \rangle \sim \exp[-\int dx \frac{T}{2} \text{Tr}(A - A^g)^2]$, we see that the effective theory is the gauged principal chiral model with a quadratic potential

$$S_{\text{eff}}[A, g] = \int dx \left(\frac{T}{e^2} \text{Tr} |\nabla g + i[A, g]|^2 - \lambda |\text{Tr} g|^2 \right). \quad (8)$$

This effective action with $\lambda = 0$ was discussed by Grignani *et al.* [9]. It is gauge invariant, $S_{\text{eff}}[A, g] = S_{\text{eff}}[A^h, hgh^\dagger]$, and has the global symmetry $S_{\text{eff}}[A, g] = S_{\text{eff}}[A, zg]$, where z is a constant element from the center of the gauge group, which for $U(N)$ is $U(1)$ and for $SU(N)$ is Z_N .

The realization of this center symmetry governs confinement [6,7]. When the symmetry is represented faithfully, the theory is in the confining phase. The Polyakov loop operator $\text{Tr} g(x)$ transforms under the center as

$\text{Tr } g(x) \rightarrow z \text{Tr } g(x)$. Thus the expectation value of the Polyakov loop operator must average to zero if the symmetry is not spontaneously broken. This expectation value is interpreted as the free energy of the system with an additional external charge in the fundamental representation of the gauge group situated at point x , $F[x, \lambda, T] = -T \ln \langle \text{Tr } g(x) \rangle$. For finite N , and $D = 3$, ideas of universality have been applied to study phase transitions with this order parameter in $SU(N)$ gauge theory [10]. The phase transition should be second order for $N = 2$ and first order for $N > 2$.

Here we analyze the effective theory (6) and (8) in the large N limit. If we rescale the coupling constant so that $\frac{e^2}{T} \equiv \frac{2\gamma}{N}$, both terms in the action (8) are of order N^2 and in the large N limit the partition function is dominated by the configuration which minimizes the action. Gauge invariance can be used to diagonalize the matrices $g_{ij}(x) = e^{i\alpha_i(x)} \delta_{ij}$. The density of eigenvalues $\rho(\theta, x) = \frac{1}{N} \sum_{i=1}^N \delta(\theta - \alpha_i(x))$ corresponding to the large N saddle point now characterizes the properties of the system. A constant density $\rho_{\text{conf}}(\theta, x) = \frac{1}{2\pi}$ realizes the center symmetry, and thus corresponds to the confining phase. A density peaked at some value of θ explicitly breaks the center symmetry, and corresponds to a deconfined phase.

If, in the general case, we consider the Fourier expansion

$$\rho(\theta, x) = \frac{1}{2\pi} + \frac{1}{2\pi} \sum_{n \neq 0} c_n(x) e^{-in\theta},$$

$$c_n(x)^* = c_{-n}(x), \quad (9)$$

the Fourier coefficients $c_n(x)$ characterize the possible deconfined phases of the theory. If one of them were nonzero, we would have in the infinite N limit $\frac{1}{N} \langle \text{Tr } g^n(x) \rangle = c_n(x)$. This would indicate that a composite of n fundamental quarks would have finite free energy and would not be confined.

In order to find the configurations of the eigenvalue density (9) that minimize the action, we shall use the collective field theory approach of Refs. [11–13]. Alternatively to the gauge fixing that we have discussed, we consider (6) in the gauge $A = 0$ which can be fixed on the open line. Then the thermodynamic problem is equivalent to unitary matrix quantum mechanics

$$Z[\lambda, T] = \int [dg] \exp \left[- \int dx \left(\frac{N}{2\gamma} \text{Tr} |\nabla g|^2 - \lambda |\text{Tr } g|^2 \right) \right]. \quad (10)$$

This model can be solved in the large N limit by the methods of collective field theory. The method is essentially based on the relation between matrix quantum mechanics and nonrelativistic fermions [14]. In the large N limit the eigenvalue density obeys a classical, saddlepoint equation which can be deduced from canonical analysis of the col-

lective field theory Hamiltonian [11]

$$H = \int d\theta \left[\frac{\gamma}{2} \rho(\theta) \left(\frac{\partial \pi}{\partial \theta} \right)^2 + \frac{\pi^2 \gamma}{6} \rho^3(\theta) \right] - \lambda \left| \int d\theta \rho(\theta) e^{i\theta} \right|^2 - \frac{\gamma}{24}, \quad (11)$$

and subsequent Wick rotation to imaginary time. Here $\Pi(\theta)$ is the variable which is the canonical conjugate of $\rho(\theta)$, so that the Poisson bracket is $\{\rho(\theta), \Pi(\theta')\} = \delta(\theta - \theta')$. The velocity of the Fermi fluid is $v(\theta) = \partial \Pi / \partial \theta$. In the equations of motion following from (11), we change $t \rightarrow ix$, $v \rightarrow -iv$ and obtain

$$\frac{\partial \rho}{\partial x} + \gamma \frac{\partial}{\partial \theta} (\rho v) = 0, \quad (12)$$

$$\frac{\partial v}{\partial x} + \gamma v \frac{\partial v}{\partial \theta} - \pi^2 \gamma \rho \frac{\partial \rho}{\partial \theta} + 2\lambda \text{Im}[e^{-i\theta} c_1(x)] = 0. \quad (13)$$

It is expected that the solution of these equations corresponding to an equilibrium state of the system is a constant $\rho(x, \theta) = \rho_0(\theta)$. At least at sufficiently low temperature or, equivalently, at sufficiently large γ , the system is in the confining phase with unbroken center symmetry, so that $\rho_0 = \rho_{\text{conf}} = 1/2\pi$. This is always a solution of the equations of motion (12) and (13) since $c_1 = 0$.

However, this solution is stable against small fluctuations only if γ is large enough. To find the spectrum of excitations in this phase, we linearize the equations of motion around ρ_{conf} . To do this, we consider the $c_n(x)$ of Eq. (9) and v infinitesimal. The resulting equation for $c_n(x)$ is

$$\left(-\nabla^2 + \frac{\gamma^2 n^2}{4} - \lambda \gamma (\delta_{n,1} + \delta_{n,-1}) \right) c_n(x) = 0, \quad n \neq 0. \quad (14)$$

At $\gamma = \gamma_c(\lambda) = 4\lambda$, the lowest eigenvalue corresponding to $n = \pm 1$ goes to zero. For smaller γ this eigenvalue is negative and the strong coupling solution is unstable with $c_{\pm 1}$ the first modes to become unstable. However, for reasons which will become clear once we consider the weak coupling phase, $\gamma_c(\lambda)$ should not be identified with the point of the deconfining phase transition.

The solution in the deconfined phase can be obtained by integration of Eq. (13) at $v = 0$. The density $\rho_0(\theta)$ can always be chosen to be an even function of θ . Thus c_1 is real, and one finds from Eq. (12):

$$\rho_0(\theta) = \frac{1}{\pi} \left[\frac{2}{\gamma} (E + 2\lambda c_1 \cos \theta) \right]^{1/2}.$$

Outside the region $[-\theta_{\text{max}}, \theta_{\text{max}}]$ with $\theta_{\text{max}} = \pi - \arccos(E/2\lambda c_1)$, the density $\rho(\theta)$ is zero.

The Fermi energy E and the constant c_1 are to be determined from the normalization condition

$$\int_{-\theta_{\text{max}}}^{\theta_{\text{max}}} d\theta \rho_0(\theta; E, c_1) = 1,$$

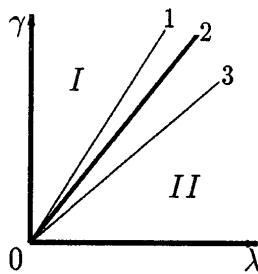


FIG. 3. The large N phase diagram of the one-dimensional model. I: strong coupling (confining) phase, II: weak coupling (deconfining) phase; 1: line on which the weak coupling phase terminates: $\gamma = \gamma_*(\lambda)$, 2: line of the first-order phase transition: $\gamma = \gamma_0(\lambda)$, 3: line of the instability of the strong coupling phase: $\gamma = \gamma_c(\lambda)$.

and the consistency condition

$$\int_{-\theta_{\max}}^{\theta_{\max}} d\theta \cos \theta \rho_0(\theta; E, c_1) = c_1$$

derived from Eq. (9). It follows from these equations that θ_{\max} tends to zero at $\gamma \rightarrow 0$ and grows with the increase of γ . Eventually it reaches π , where the weak coupling phase terminates, because the eigenvalue distribution begins to overlap with itself due to 2π periodicity. At the critical point $E_* = 2\lambda c_{1*}$, the normalization and consistency integrals can be done explicitly. We find that $c_{1*} = 1/3$ and $\gamma_*(\lambda) = \frac{128}{3\pi^2} \lambda \approx 4.324 \lambda$.

We obtain the following picture of the deconfining phase transition (Fig. 3). The weak and strong coupling phases can coexist, because $\gamma_c(\lambda) < \gamma_*(\lambda)$, although the region, where both phases are stable is very narrow, since $\gamma_c(\lambda)$ and $\gamma_*(\lambda)$ are numerically close to each other. The phase transition is of the first order and takes place at some $\gamma_0(\lambda)$ between $\gamma_c(\lambda)$ and $\gamma_*(\lambda)$. At the point of the phase transition the free energies of both phases are equal to each other. Substituting $\rho_0(\theta)$ into Eq. (11) one can find the free energy per unit volume, to leading order in the large N limit,

$$\frac{F}{N^2} = \begin{cases} 0, & \text{in the confining phase,} \\ \frac{1}{3}E - \frac{1}{3}\lambda c_1^2 - \frac{\gamma}{24}, & \text{in the deconfined phase.} \end{cases} \quad (15)$$

The equations determining the critical line can be solved numerically to obtain $\gamma_0(\lambda) = 4.219\lambda$.

The model which we have considered here is adjoint QCD in the limit where the particles are heavy. The fugacity parameter can be computed from a one-loop diagram as $\lambda = \sqrt{\frac{mT}{2\pi}} e^{-m/T}$, the exponential being simply the Boltzmann weight of a particle with mass m . It is assumed that $m \gg T$ for classical statistical mechanics to be applicable and $m \gg e$ to suppress pair production. Our results indicate that the phase transition is of first order with critical line approximately given by the equation $\frac{e^2 N}{2T} \approx 4.2 \sqrt{\frac{mT}{2\pi}} e^{-m/T}$. There exists a region of parameters in which this equation has a solution and the conditions of applicability of our simplified model are satisfied.

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