Does Chaotic Mixing Facilitate $\Omega < 1$ Inflation?

Neil J. Cornish,[1] David N. Spergel,[2,3] and Glenn D. Starkman[1]

[1]*Department of Physics, Case Western Reserve University, Cleveland, Ohio 44106-7079*
[2]*Princeton University Observatory, Princeton, New Jersey 08544*
[3]*Department of Astronomy, University of Maryland, College Park, Maryland 20742*

Yes, if the Universe has compact topology.   [S0031-9007(96)00617-5]

Inflation is currently the most elegant explanation of why the Universe is old, large, nearly flat, homogeneous on large scales, and structured on small scales [1].

One of the most robust predictions of the inflationary scenario, particularly Linde's chaotic inflationary model, is that $|\Omega - 1|$ is exponentially suppressed and is nearly zero. This prediction appears to be contradicted by determinations of the Hubble constant, $h \simeq 0.75$, observations of large-scale structure that imply that $\Omega_{nr} h \sim 0.25$ [2], and stellar ages that appear to exceed the age of the Universe for these parameters [3]. Here, Ω and Ω_{nr} denote the ratio of the total energy density to the closure density and the ratio of the energy density in nonrelativistic particles to the closure density. This contradiction between theory and observation has motivated theorists to evoke a cosmological constant and to explore the possibility of an open inflationary universe [4,5].

One of the weaknesses of the inflationary paradigm is the problem of initial conditions for inflation: the preinflationary universe must be somewhat old, somewhat large, and somewhat homogeneous [1,6]. In Linde's chaotic inflation model [1] and eternal inflation model [7] this is natural, but inflation must occur at the Planck scale. These initial condition requirements are even more severe in $\Omega < 1$ inflationary models: If the Universe does not inflate enough to appear flat, then it does not inflate enough to appear homogeneous [8].

One solution is to have two inflationary epochs: the first inflation erases all inhomogeneities and ends with the nucleation of an $\Omega < 1$ bubble, which inflates by exactly 69 e-foldings to produce the observed Universe [4]. The naturalness of such models has been discussed extensively [5]. Besides the usual fine tuning of the inflaton potential to control the amplitude of density fluctuations, and the necessity, in any $\Omega < 1$ universe model, of arranging that Ω is relatively close to unity today, additional tuning is necessary to get both inflations out of one set of dynamics. Open-inflation models have been constructed [5] that avoid this latter fine tuning by using more than one inflaton field.

In this Letter, we propose another solution to the problem of preinflationary homogeneity: if the Universe is compact, then during the preinflationary period there is typically sufficient time for chaotic mixing to smooth out primordial fluctuations. Gradients in the energy-density are reduced as $e^{-\kappa d}$, where κ is the Kolmogorov-Sinai (KS) entropy of the flow, and d is the distance the flow travels [9]. This Letter explores this homogenization process, outlines why small, compact, negatively curved universes are the most natural, and concludes with a discussion of the implications of living in such a universe.

There are several physical and philosophical motivations for considering compact universes. Einstein and Wheeler advocate finite universes on the basis of Mach's principle [10]. Others argue that an infinite universe is unaesthetic and wasteful [11], because anything that can happen does happen, and an infinite number of times.

Quantum cosmologists have argued [12] that small volume universes have small action and are therefore more likely to be created. More intuitively, it is more difficult to produce a large universe. Finally, a common feature of many quantum theories of gravity is the compactification of some spacelike dimensions. This suggests a dimensional democracy, in which all dimensions are compact, and geometry distinguishes the large ones from the small ones. Positively curved dimensions remained at or collapsed to Planck scales in a Planck time, while negatively curved dimensions grew to macroscopic proportions.

The idea of topological or chaotic mixing is also not new [13]. Chaotic mixing has been considered as an alternative to inflation [14,15]. However, chaotic mixing does not solve the flatness, age, and monopole problems, and for it alone to solve the horizon problem the topology scale today would have to be unacceptably small. By marrying chaotic mixing to inflation, we retain the benefits of inflation and solve the preinflationary initial-value problem, and in particular the large-scale homogeneity problem for $\Omega < 1$ universes.

Most of the scant attention to nontrivial compact topologies in cosmology has focused on the simplest nontrivial topology of the flat geometry: a cube with opposite sides identified, i.e., a 3-torus. While the Universe may be truly flat ($\Omega \equiv 1$, not just $|\Omega - 1| \ll 1$), flat manifolds are measure zero in the set of possible 3-manifolds. Moreover, in flat universes the geometry sets no scale, so the size of the fundamental cell of the topology is arbitrary. It would be an unnecessary coincidence for that scale to be of the order of horizon size today. Positively curved universes are inherently compact; however, they typically recollapse on order the Planck time, 10^{-42} s. Either inflation must begin at the Planck time, as it does in

Linde's chaotic inflation models [1], or the Universe must be unnaturally flat in order to grow cold enough to allow inflation to begin.

It has been conjectured that most 3-manifolds are topologically equivalent to 3-manifolds of constant negative curvature [16]. This greatly simplifies the description and classification of 3-manifolds. The universal covering space of the constant negative curvature geometry is H^3. The classification of topologies of H^3 is then isomorphic to finding the discrete subgroups of the group of metric-preserving transitive motions of H^3. This group is isomorphic to the proper Lorentz group $PSL(2, C)$, which is clear if we think of H^3 as a 3-sphere of imaginary radius embedded in four-dimensional flat space. Of particular interest are torsion-free subgroups as these describe compact 3-manifolds.

While there are an infinite number of compact hyperbolic 3-manifolds, they are classifiable in terms of their volumes, just as 2-manifolds are classifiable by genus. It has been shown [17] that the volume of any compact hyperbolic 3-manifolds is bounded below by $V_{\min} = 0.000\,82 R_{\mathrm{curv}}^3$, where R_{curv} is the radius of curvature. Many explicit examples have been constructed with small volumes. Some relatively simple topologies have been constructed by identifying the faces of the four hyperbolic analogs of the Platonic solids, the hexahedron, icosahedron, and two dodecahedra [18]. These typically have volumes in the range $(4-8)R_{\mathrm{curv}}^3$, but other examples have volumes as small as $0.94 R_{\mathrm{curv}}^3$ [19]. For our purposes, the volume of the topology is far more important than the specific form of its identification group.

We will begin by considering preinflationary universes that are perturbations around a homogeneous and isotropic Friedmann-Robertson-Walker (FRW) solution. The Friedmann equation,

$$\left(\frac{da}{d\eta}\right)^2 = \frac{8\pi G a^4}{3}\left[\left(\frac{a_0}{a}\right)^4 \rho_i^{\mathrm{rad}} + \left(\frac{a_0}{a}\right)^2 \rho_i^{\mathrm{curv}} + \rho^{\mathrm{vac}}\right],$$

(1)

governs the evolution of this Universe, where a is the scale factor and η is the conformal time. We choose a_0, the scale factor when the Universe is nucleated, to be unity without loss of generality. We have rewritten the usual curvature term in terms of an energy density, $\rho_i^{\mathrm{curv}} = 3M_{\mathrm{Pl}}^2/8\pi R_{\mathrm{curv}}^2$, where R_{curv} is the comoving curvature scale. As we have written explicitly, the vacuum energy density remains constant as the Universe expands, the radiation energy density drops as a^{-4}, and the curvature energy density drops as a^{-2}. We have neglected the Casimir energy associated with the finite volume, but it typically behaves like radiation.

What do we expect for the properties of a "typical" preinflationary universe? Since the only characteristic scale in quantum gravity is the Planck scale, $M_{\mathrm{Pl}}^{-1} = \sqrt{\hbar c/G}$, we imagine the typical Universe will start with an initial volume, $V_{\mathrm{init}} = C^3/M_{\mathrm{Pl}}^3$, an initial comoving

curvature scale, $R_{\mathrm{curv}} = C/\alpha M_{\mathrm{Pl}}$, an initial radiation density, $\rho_i^{\mathrm{rad}} = \gamma^2 M_{\mathrm{Pl}}^4$, and an initial vacuum energy, $\rho^{\mathrm{vac}} = \lambda M_{\mathrm{Pl}}^4/4$. For compact negatively curved manifolds, $V \equiv \alpha^3 R_{\mathrm{curv}}^3$ where $\alpha > 0.0936$.

We expect $C \gtrsim 1$, if only so our classical treatment of the geometry makes sense. If nucleation of large universes is suppressed [12], C should not be too big.

The value of γ is also uncertain. If $\rho_i^{\mathrm{rad}} \lesssim M_{\mathrm{Pl}}^4$, then $\gamma^2 \sim 1$; if the initial energy in radiation is $\lesssim M_{\mathrm{Pl}}$, then $\gamma^2 C^3 \lesssim 1$; finally, if the initial energy times the light-crossing time is ~ 1 (á la Heisenberg), then $\gamma^2 C^4 \lesssim 1$. The latter estimate is probably the most appropriate for a finite volume universe. Interpreting the curvature as a source of energy density, $\rho_i^{\mathrm{curv}} = 3\alpha^2 M_{\mathrm{Pl}}^4/8\pi C^2$, we can argue similarly that $\alpha C \lesssim \sqrt{8\pi/3}$.

In order for inflation to be consistent with the COBE detections of large scale temperature fluctuations, λ must be of order 10^{-15}, where the actual value depends on the details of the inflaton potential [1]. In the "new inflation" scenario, this implies that initially ρ^{vac} makes a minor contribution to the total energy density of the Universe. Thus the Universe begins either radiation dominated if $\gamma > \sqrt{3/8\pi}\,(\alpha/C)$, or curvature dominated otherwise. During the radiation-dominated period,

$$\eta = \sqrt{\frac{3}{8\pi}}\,\frac{(a-1)}{\gamma M_{\mathrm{Pl}}}.$$

(2)

The Universe quickly becomes curvature dominated at $a_{RC} = \sqrt{8\pi/3}\,(C\gamma/\alpha)$. During the curvature-dominated epoch,

$$\eta - \eta_{RC} = R_{\mathrm{curv}} \ln\left[\frac{a}{a_{RC}}\right].$$

(3)

Finally, when $\rho^{\mathrm{curv}} = \rho^{\mathrm{vac}}$ at $a_{CV} = \sqrt{3\alpha^2/2\pi\lambda C^2}$, the Universe becomes vacuum dominated and

$$\eta - \eta_{CV} = \frac{3}{8\pi\lambda M_{\mathrm{Pl}}^2}\left(\frac{1}{a_{CV}} - \frac{1}{a}\right).$$

(4)

Consider the evolution of the primordial fluctuations during the radiation- and the curvature-dominated phase. The fluctuations can be expanded as a sum of eigenmodes of the Laplacian on H^3 that satisfy the periodicity conditions of the identification group. If the fluctuations are in a strongly coupled plasma, they will propagate as acoustic waves at a sound speed near the relativistic value, $c_s = 1/\sqrt{3}$. In an open, curvature-dominated universe, these modes oscillate as free waves that decay only through nonlinear effects [20]. Classically, the fluctuations in a weakly interacting field, such as the inflaton, behave as free waves propagating in the expanding background.

All fluctuations will, however, be suppressed by chaotic mixing. It is well known to mathematicians [21] that geodesic flows on a compact negatively curved manifold (CNCM) are ergodically mixed—bundles of trajectories are stretched and folded like bakers' dough. A CNCM

has an infinite number of unstable eigenmodes. If initially just one eigenmode is excited, the instability quickly ensures that all nearby eigenmodes are excited, and the initial energy spread between them. Since the density of eigenmodes grows exponentially with wavelength, the system rapidly approaches a state indistinguishable from the homogeneous background. That is, the chaotic mixing acts as an effective (noncollisional) dissipative mechanism. Given some initial distribution of any propagating field, the multipoint correlations decay as $e^{-\kappa d}$, where d is the distance a mode of the field has propagated and κ is the KS entropy of the flow. For a CNCM of volume V, $\kappa \simeq V^{-1/3}$. ($L = V^{1/3}$, the "topology scale," is approximately the distance across the fundamental cell.) This description applies exactly to the cosmological situation, where η (or $c_s\eta$) is the comoving distance traveled by a mode and $\kappa \simeq 1/\alpha R_{\text{curv}} \ln 2$ is the effective KS entropy [15].

One may also be concerned that studying classical flows is not appropriate for understanding the behavior of inherently quantum mechanical fluctuations, especially for the weakly coupled inflaton. This concern is unfounded. Quantum chaos on CNCMs is known to exhibit the same ergodic mixing seen in the classical description [22]. The positive KS entropy is manifested as an exponential growth in the number of low energy modes, and the instability leads to the exponential decay of the amplitude of any given mode. In the limit of perfect mixing, the wave functions become completely random and uncorrelated, in contrast to the ordered wave functions found in integrable quantum systems.

The mixing can be understood as follows: Initially the horizon expands to encompass many copies of the fundamental cell (in the covering space). During this time we expect substantial chaotic mixing. Once the Universe becomes vacuum dominated, however, the horizon shrinks rapidly below the topology scale, and we expect little or no further mixing. Postinflation, the horizon scale grows once again. Today it may encompass several fundamental domains, leading to a new period of chaotic mixing. This mixing would give an additional suppression of the large-scale inhomogeneities, which we neglect below. Postrecombination mixing has been considered, and ultimately rejected [23], as a self-contained explanation for the isotropy of the cosmic microwave background radiation (CMBR).

From our considerations of the initial state of the Universe, we expect $\rho_i^{\text{rad}}/\rho_i^{\text{curv}} < 8\pi/3\alpha^2$. For $\alpha \gtrsim 1$, almost the entire preinflationary evolution will be curvature dominated. Primordial density fluctuations are thus reduced by a factor of

$$\frac{(\delta\rho/\rho)_f}{(\delta\rho/\rho)_i} = \exp[-\kappa c_s \Delta\eta] \simeq \left(\frac{2\pi\lambda}{3}\frac{C^2}{\alpha^2}\right)^{c_s/2\alpha\ln 2}. \quad (5)$$

For acoustic modes $c_s \simeq 1/\sqrt{3}$. Fluctuations in the inflaton are more strongly suppressed, since $c_s = 1$. The level of homogeneity necessary for inflation to begin is quite mild; however, to explain the isotropy of the CMBR on large scales, evolution during the pre-exponential-inflation phase must suppress temperature fluctuations to at most 10^{-5}. Since the total energy in radiation is initially only $C^3\gamma^2 M_{\text{Pl}}$, and since radiation redshifts, only the fluctuations in the inflaton need be so severely suppressed. These 10^{-5} temperature fluctuations translate into inhomogeneities in the inflaton energy density of $\lesssim 8 \times 10^{-5}(1/\Omega - 1)$ for modes of wavelength longer than the curvature scale [4]. Starting from $\delta\rho/\rho \simeq 1$ initially (standard nonlinear damping will quickly cut larger fluctuations down to this size), and saturating the bound $\alpha C \lesssim \sqrt{8\pi/3}$, we find that sufficient mixing can be achieved if $\alpha \lesssim 2.7$ for $\Omega = 0.3$, and $\alpha \lesssim 3.5$ for $\Omega = 0.1$ for $\lambda = 10^{-15}$. Thus, small universes, whose topology scale is comparable to the curvature scale, will have sufficient time to erase primordial fluctuations. When the vacuum energy starts to dominate and inflation begins, the horizon shrinks below the topology scale and mixing rapidly terminates. Therefore, we require that the inflaton is already rolling by the time the mixing stops, otherwise quantum fluctuations in the inflaton can lead to large temperature fluctuations. The inflaton potential must be of a form that ensures the inflaton comes out of the curvature-dominated epoch rolling at its full slow-roll velocity [5].

The exact value of our limit on α relies heavily on the value of the KS entropy, which we have approximated by its isotropic average. It relies more weakly on the value of λ, and thus the model used to extract λ from the fluctuation spectrum. For example, for a pure $(\lambda/4)\phi^4$ potential we would have found $\alpha < 2.4$ for $\Omega = 0.3$, $\alpha < 2.6$ for $\Omega = 0.1$. Universes with somewhat higher values of α should be investigated individually.

So far, we have restricted ourselves to a constant-curvature background with perturbations. This was so we could perform an analytic calculation. We believe that physically this simplification is unnecessary. The topology of any 3-manifold is conjectured to be equivalent to a manifold formed from homogeneous primitives [16]. It has been shown that most [16] 3-manifolds consist of one negatively curved primitive. Because compact H^3 topologies lead to chaotic mixing, we conjecture that most 3-geometries will evolve to a homogeneous FRW of negative curvature.

In summary, if the Universe has negative curvature and compact topology, the most generic 3-manifold, then chaotic mixing smooths out primordial inhomogeneities. We have shown that for compact negatively curved universes with volumes on the order of a few Planck volumes at nucleation, sufficient chaotic mixing will occur to solve the large-scale-inhomogeneity, inflationary-initial-value problem. Thus, unlike in infinite negatively curved universes, two stages of inflation are not required. We have also argued that this mechanism will operate in general geometries, causing them to evolve to a FRW geometry. In the absence of observable curvature, this

mechanism can solve the initial-value problem of inflationary models, but is not expected to have observational consequences. More optimistically, if astronomers have detected the curvature of the Universe, they may soon detect the effects of its finite size.

As mentioned above, most of the attention to compact topologies in cosmology has focused on the 3-torus. The earliest works looked for objects in the sky which could be seen in more than one direction, much as you see yourself in a hall of mirrors. By using objects that evolve relatively slowly, lower limits on the topology scale of order 200 MpC were set [24].

The effects of compact topology may be more easily seen in the CMBR. In flat topologies, modes of wavelength larger than the identification scale (up to 6 times the topology scale, depending on the topology) do not exist, reducing the amplitude of fluctuations on large angular scales. This has been searched for in the case of the 3-torus [25]. In negatively curved compact models there is no such cutoff. Some authors have proposed searching for periodicity in the pattern of CMBR hot and cold spots [26]. More promising, we believe, is to recognize that in these small universes the CMBR will be identified at the intersections of the surface of last scattering (SLS), as seen by different "copies" of the observer (in the covering space). Here, small means that the diameter of the SLS is smaller than the topology scale. Since the SLS is a sphere, these intersections will be circles, regardless of the background geometry or topology. Thus fluctuations in the CMBR would be correlated on circles of the same radii centered on different points in the sky. The existence of these correlated circles will allow us to search for the existence of topology, independent of the particular topology in question [27]. The COBE/DMR4 data set [28] is currently being analyzed to search for this signal, but its signal-to-noise ratio and angular resolution are probably inadequate. However, data from CMBR satellites planned for the next decade should allow us to decide definitively if there is topology on the scale of the observed Universe.

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