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Perturbative Expansion for Coherence Loss

Ji Il Kim and M. C. Nemes
Departamento de Física, ICEX, Universidade Federal de Minas Gerais, C.P. 702, 30161-970 Belo Horizonte, MG, Brazil

A. F. R. de Toledo Piza
Instituto de Física, Universidade de São Paulo, C.P. 66318, 05389-970 São Paulo, SP, Brazil

H. E. Borges
Centro Federal de Educação Tecnológica de Minas Gerais, Av. Amazonas 7675, CEP30510-000 Belo Horizonte, MG, Brazil
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We construct a generally applicable short-time perturbative expansion for coherence loss. Successive terms of this expansion yield characteristic times for decorrelation processes involving successive powers of the Hamiltonian. The second order results are sufficient to precisely reproduce expressions for "decoherence times" obtained in the literature by much more involved and indirect methods. Examples illustrating the influence of initial conditions and the need to evaluate higher order terms are given in the context of the Jaynes-Cummings model. It is shown that, in this case, the short-time decoherence behavior can probe the importance of antiresonant contributions. [S0031-9007(96)00590-X]

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The study of open quantum systems and/or subsystems has recently attracted the attention of physicists from very different areas: cosmology [1], condensed matter [2], quantum optics [3], particle physics [4], as well as of theorists working on the fundamentals of the quantum measurement process [5]. The problem can be stated very generally by considering several interacting subsystems and asking for the looks of the effective dynamics of one such subsystem. Generic, exact answers within the standard framework of quantum mechanics have been given before [6]. Recent experimental developments [7] as well as the analysis of models related to them [8] now indicate, however, that the specific knowledge of the (often very short) time scale for the onset of decoherence processes may be of considerable value. In order to meet such demand we develop here a short-time perturbative scheme to extract decorrelation time scales from the in general highly nonlinear effective dynamics of open quantum subsystems. Our results are generally applicable to situations in which the subsystem of interest appears as part of a larger, closed Hamiltonian system. They are based on a hierarchical analysis of the short-time dynamics of *intersubsystem* correlation processes which bears a strong resemblance in spirit to ordinary time-dependent perturbation expansions.

We consider the general case of a dynamically closed (i.e., autonomous) quantum system which is described as being composed of two interacting subsystems, so that the full Hamiltonian is written as a sum of three terms

$$H = H_{01} + H_{02} + H_{\rm int} \equiv H_0 + H_{\rm int}, \qquad (1)$$

the last of which represents the interaction between the subsystems, while H_0 describes their bare dynamics. Note that no *a priori* limitation is being imposed on the nature or complexity of the subsystems. In particular, all current models involving quantum systems coupled to dynamically implemented reservoirs (e.g., [9]) fit into the above characterization, the same being true all the way to very simple systems (e.g., models involving coupled spins [10] or the Jaynes-Cummings model for the interaction of atoms with radiation [8,11,12]). The state of the system is described in full generality in terms of a density operator $\rho(t)$ evolving in the Schrödinger picture as

$$\rho(t) = e^{-iHt}\rho(0)e^{iHt}, \tag{2}$$

where $\rho(0)$ stands for the initial density operator (at $t = 0$). We consider then the reduced density for one of the subsystems (say, subsystem 2)

$$\rho_2(t) = \mathrm{Tr}_1\{\rho(t)\}, \tag{3}$$

where Tr_1 denotes the trace over the degrees of freedom of subsystem 1.

A simple and direct measure of the degree of decoherence in subsystem 2 is provided by the "idempotency defect" $\delta(t)$ of the reduced density $\rho_2(t)$ [13]. This is the quantity written in the equation below, where it is furthermore subjected to a short-time power series expansion:

$$\begin{aligned}\delta(t) &\equiv \mathrm{Tr}_2\{\rho_2(t) - \rho_2^2(t)\} \\ &= 1 - \mathrm{Tr}_2\left(\rho_2^2(0) + t\frac{d\rho_2^2}{dt}(0) + \frac{t^2}{2!}\frac{d^2\rho_2^2}{dt^2}(0) + \cdots\right) \\ &\equiv \delta_0 - \frac{t}{\tau_1} - \frac{t^2}{\tau_2^2} - \cdots .\end{aligned} \tag{4}$$

Here we use the normalization condition $\mathrm{Tr}_2\rho_2(0) = 1$ and the fact that it is preserved in time. The coefficients of this expansion are readily obtained from Eq. (2). Up to order t^2 they read explicitly

$$\delta_0 = 1 - \mathrm{Tr}_2\{\rho_2^2(0)\}, \tag{5}$$

$$\begin{aligned}\frac{1}{\tau_1} &= 2\mathrm{Tr}_2\{\rho_2(0)\dot{\rho}_2(0)\} \\ &= 2i\mathrm{Tr}_2\{\rho_2(0)\,\mathrm{Tr}_1[\rho(0), H]\} \\ &\equiv 2i\langle \mathrm{Tr}_1[\rho(0), H]\rangle_{\rho_2(0)},\end{aligned} \tag{6}$$

and

$$\begin{aligned}\frac{1}{\tau_2^2} &= \mathrm{Tr}_2\{\dot{\rho}_2^2(0) + \rho_2(0)\ddot{\rho}_2(0)\} = -\mathrm{Tr}_2\{(\mathrm{Tr}_1[\rho(0), H])^2 + \rho_2(0)\,\mathrm{Tr}_1[[\rho(0), H], H]\} \\ &\equiv -\mathrm{Tr}_2\{(\mathrm{Tr}_1[\rho(0), H])^2\} - \langle \mathrm{Tr}_1[[\rho(0), H], H]\rangle_{\rho_2(0)},\end{aligned} \tag{7}$$

where the symbol $\langle\cdots\rangle_{\rho_2(0)}$ stands for an average value over the state $\rho_2(0)$. The above expressions constitute the first few terms in the short-time expansion of the time development of those quantum processes which are specifically related with coherence loss (decoherence) within the subsystems. These processes involve the dynamics of intersubsystem correlations, which cause the time evolution of the reduced densities to be nonunitary. In fact, a change of $\delta(t)$ is associated to a change of the eigenvalues of the reduced density [6]. The coefficients of the expansion furnish characteristic times τ_n associated with correlation processes involving H_{int} to order n. Note that they can be evaluated once the Hamiltonian is specified and the initial state of the composite system is given. It may be noted that the double commutators such as those appearing in Eq. (7) have been discussed in the context of irreversible processes where they appear in connection with dynamical semigroups related to master equation time evolution [9]. Equations (6) and (7) involve all the physical ingredients which may contribute to the time development of quantum correlations to first and second order in H_{int}. The same expressions can be obtained from the exact dynamics of the eigenvalues of the reduced density matrix, as given in Ref. [6].

In the frequently considered special case in which the two subsystems are initially uncorrelated, i.e., when the initial density $\rho(0)$ factors as $\rho(0) = \rho_1(0) \otimes \rho_2(0)$, the above expressions simplify considerably. We get

$$\delta_0 = 1 - \mathrm{Tr}_2\{\rho_2^2(0)\}, \qquad \frac{1}{\tau_1} = 0, \tag{8}$$

and

$$\begin{aligned}\frac{1}{\tau_2^2} &= \mathrm{Tr}_2\{\rho_2(0)\,([[\rho_2(0), \langle H\rangle_{\rho_1(0)}], \langle H\rangle_{\rho_1(0)}] \\ &\quad - \mathrm{Tr}_1[[\rho_1(0)\rho_2(0), H], H])\},\end{aligned} \tag{9}$$

where $\langle\cdots\rangle_{\rho_1(0)}$ again denotes an average taken with respect to the density $\rho_1(0)$. If furthermore $\rho_1(0) = |\psi_1\rangle\langle\psi_1|$ and $\rho_2(0) = |\psi_2\rangle\langle\psi_2|$ are idempotent densities (pure quantum states), $\delta_0 = 0$ and Eq. (9) simplifies further to

$$\begin{aligned}\frac{1}{2\tau_2^2} &= -\langle\psi_1\psi_2|H|\psi_1\psi_2\rangle^2 + \langle\psi_1|\langle H\rangle_2^2|\psi_1\rangle \\ &\quad + \langle\psi_2|\langle H\rangle_1^2|\psi_2\rangle - \langle\psi_1\psi_2|H^2|\psi_1\psi_2\rangle,\end{aligned} \tag{10}$$

where now $\langle H\rangle_n$ stands for $\langle\psi_n|H|\psi_n\rangle$.

Equation (8) shows in general that for two initially uncorrelated subsystems decoherence processes evolve at least quadratically in time for short times. In order to gain further insight into the content of these expressions we now show that Eq. (9) corresponds precisely to the well-known expression for the decoherence time obtained for the popular model of Refs. [2,14] through the use of a Fokker-Planck equation or directly from the Feynman-Vernon [15] influence functional. This model consists of a harmonic oscillator (to be considered as subsystem 2) linearly coupled to a heat bath (subsystem 1) made also of harmonic oscillators, the Hamiltonian being written as

$$\begin{aligned}H &= \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2x^2 + \sum_k\left(\frac{p_k^2}{2m_k} + \frac{1}{2}m_k\omega_k^2x_k^2\right) \\ &\quad + x\sum_k c_kx_k,\end{aligned} \tag{11}$$

where c_k are the coupling constants. If one considers the initial state

$$\rho(0) = \left(\prod_k \frac{e^{-\beta H_k}}{Z_k}\right) \otimes |\psi\rangle\langle\psi| , \qquad (12)$$

where β is the usual Boltzmann factor, $H_k(Z_k)$ is the Hamiltonian (partition function) of the kth oscillator in the heat bath, and $|\psi\rangle\langle\psi|$ the initial state of the oscillator with frequency w_0, from Eq. (10) one gets

$$-\frac{1}{\tau_2^2} = 2\Delta^2 x\left[\sum_k \frac{c_k^2}{2m_k\omega_k}\coth\left(\frac{\beta\hbar\omega_k}{2}\right)\right], \qquad (13)$$

where $\Delta^2 x$ is the variance of the position coordinate of the ω_0 oscillator in the state $|\psi\rangle$.

Now, with the usual Ohmic dissipation assumptions [3], i.e.,

$$\sum_k \to \int_0^\infty d\omega D(\omega) \quad \text{with} \quad \frac{D(\omega)c_\omega^2}{\omega^2 m_\omega} = \begin{cases} 0, \omega > \frac{1}{\tau_2} \\ \frac{\eta}{\pi}, \omega < \frac{1}{\tau_2} \end{cases}, \qquad (14)$$

where $D(\omega)$ is the density of modes of the heat bath and η related to a friction coefficient. It is easy to show that, for $\beta\hbar\omega \ll 1$, this leads to the standard result [2]

$$|\tau_2| = \frac{\hbar^2\beta}{2\eta\ \Delta x^2} . \qquad (15)$$

We turn now to a different example, of relevance in quantum optics: the Jaynes-Cummings model whose Hamiltonian including both resonant and antiresonant contributions is given by

$$H = \hbar\omega_F(a^\dagger a + \tfrac{1}{2}) + \tfrac{1}{2}\epsilon\sigma_z + g(a^\dagger\sigma_- + a\sigma_+) + g'(a^\dagger\sigma_+ + a\sigma_-) , \qquad (16)$$

where a and $a^\dagger$ are the field boson creation and annihilation operators, respectively, $\omega_F(\epsilon)$ is the field (atomic) frequency, and the σ matrices the usual Pauli matrices with $\sigma_\pm = \sigma_x \pm i\sigma_y$. We consider the case of an initially factorized density

$$\rho(0) = |\mu\rangle\langle\mu| \otimes |\alpha\rangle\langle\alpha| , \qquad (17)$$

where $|\alpha\rangle$ stands for a coherent state of the field, considered to be subsystem 1, $a|\alpha\rangle = \alpha|\alpha\rangle$ [16], and $|\mu\rangle$ is the most general state for the spin-1/2 subsystem written as

$$|\mu\rangle = \frac{1}{(1 + |\mu|^2)^{1/2}}(|-\rangle + \mu|+\rangle) , \qquad (18)$$

the states $|\pm\rangle$ being normalized eigenvectors of σ_z. In this case the second order decoherence time scale is given by

$$-\frac{1}{\tau_2^2} = \frac{g^2}{2}(1 + \xi)^2 + \frac{g'^2}{2}(1 - \xi)^2 - gg'(1 - \xi^2)\cos^2\theta , \qquad (19)$$

where the parameters ξ and θ reparametrize the initial state of the two level subsystem according to the definition

$$\mu = \sqrt{\frac{1 + \xi}{1 - \xi}}\, e^{i\theta}, \qquad -1 \le \xi \le 1 . \qquad (20)$$

We remark first that τ_2 is independent of the displacement α characterizing the coherent state of the field. Also the well-known result for τ_2 in the so-called rotating wave approximation (RWA, $g' = 0$) and assuming the spin-1/2 initial state to be $|+\rangle$ ($\xi = 1$) is obtained from Eq. (19):

$$\left|\frac{1}{\tau_2^{\mathrm{RWA}}}\right| = \sqrt{2}g . \qquad (21)$$

This reflects the well-known result that for the atom-initial condition $|+\rangle$ the RWA of the Jaynes-Cummings model predicts Rabi oscillations with decreasing amplitude. The amplitude has in fact an envelope which can be represented for short times by the Gaussian $e^{-g^2t^2}$ [12]. Had we started with the atomic state $|-\rangle$, we would get that the envelope is also a Gaussian, but with second order decay time proportional to the antiresonant coupling g':

$$\left|\frac{1}{\tau_2}\right| = \sqrt{2}g' \quad \text{for } \rho(0) = |-\rangle\langle-| \otimes |\alpha\rangle\langle\alpha| . \qquad (22)$$

An interesting situation occurs when, perhaps more realistically, $g = g'$: If we take the initial condition $\xi = 0$, $\theta = 0$,

$$|\mu\rangle\langle\mu| = \frac{1}{2}(|+\rangle + |-\rangle)(\langle+| + \langle-|) \qquad (23)$$

we get $1/\tau_2 = 0$. In such a situation (as in the case of any perturbative expansion) we have to examine the higher order contributions. For the specific initial spin state Eq. (23) and with the field in a coherent state we get that the lowest contributing order is

$$\frac{1}{\tau_4^4} = -\frac{1}{4}\langle h_1^2\rangle_{\rho_1(0)}\langle[[H_{02}, h_2], [[H_2, h_2], \rho_2(0)]]\rangle_{\rho_2(0)} - \frac{1}{4}\langle h_1\rangle_{\rho_1(0)}^2\langle[[H_{02}, h_2], \rho_2(0)]^2\rangle_{\rho_2(0)} , \qquad (24)$$

where we have rewritten the Hamiltonian H as

$$H = H_{01} + H_{02} + gh_1h_2 \qquad (25)$$

with $H_{01} = \hbar\omega_F(a^\dagger a + \frac{1}{2})$, $H_{02} = \frac{1}{2}\epsilon\sigma_z$, $h_1 = (a + a^\dagger)$, and $h_2 = \sigma_x$. When the initial density $\rho_1(0)$ is given by Eq. (23) and $\rho_2(0) = |\alpha\rangle\langle\alpha|$ we find

$$-\left(\frac{1}{\tau_4}\right)^4 = 2\epsilon^2 g^2. \qquad (26)$$

These results immediately imply that the antiresonant terms tend to change the shape of the envelope of Rabi oscillations of the atomic inversion for short times. Also, in the hypothetical situation where we could control the parameters ϵ and g our results indicate that by carefully choosing the initial condition, special states, namely, the eigenstates of the σ_x operator, could be obtained which would lose coherence at a much slower rate than other.

We mention finally that energy dissipation rates in an open subsystem can be estimated in the same spirit of Eq. (4) by similarly expanding the average subsystem energies

$$\frac{E_i(t)}{E_i(0)} = \frac{\mathrm{Tr}\{H_{0i}\rho(t)\}}{\langle H_{0i}\rangle_{\rho(0)}}, \qquad i = 1, 2, \tag{27}$$

where the notation of Eqs. (1) and (2) has been used. It should be kept in mind that the sum $E_1(t) + E_2(t)$ differs from the total (conserved) mean energy $\mathrm{Tr}\{H\rho(t)\}$ by the interaction energy $\mathrm{Tr}\{H_{\mathrm{int}}\rho(t)\}$.

In conclusion, we implemented a very straightforward expansion procedure which allows one to estimate decorrelation times for an open subsystem of a larger, closed Hamiltonian quantum system, for *any* given initial condition of this larger system. In particular, it allows for a quantitative evaluation both of the decorrelation propensities of different initial states for a given dynamics and of the decorrelation effects of different intersubsystem couplings. A crucial point for the given implementation consists of selecting quantities, associated to the subsystem of interest, whose time evolution shows a highly selective sensitivity to the coherence properties of its state. We use the old-fashioned "idempotency defect" [13] for the purpose of studying decoherence. Analogous procedures can be devised also for other kinetic properties, and we briefly indicate in Eq. (27) quantities which are relevant in connection with energy dissipation.

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