Mesoscopic Fluctuations of Elastic Cotunneling

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(Received 6 May 1996)

We study mesoscopic fluctuations of the conductance through a quantum dot at the wings of the Coulomb blockade peaks. At low temperatures, the main mechanism of conduction is elastic cotunneling. The conductance strongly fluctuates with an applied magnetic field. The magnetic correlation field is shown to be controlled by the charging energy, and the correlation function has a universal form. The distribution function for the conductance obtained analytically shows a nontrivial crossover between the orthogonal and unitary ensembles. [S0031-9007(96)01096-4]

PACS numbers: 73.23.Hk, 03.65.Sq, 73.40.Gk

Statistical theory of electron eigenfunctions in complex systems was developed in the early 1950s [1] for the description of the absorption spectra of large atoms. Soon it was realized that the same random matrix (RMT) approach can be applied to the excitation spectrum of small metallic grains [2]. An important distinction should be made between the excitations that preserve the number of electrons in the grain and the excitations that change this number. This distinction is most easily seen in the case where the introduction of an additional electron causes charging of the grain and, thus, is associated with the charging energy E_c. In this case, the addition spectrum shows the gap of the width E_c, whereas the spectrum of electron-hole excitations is controlled by typically a much smaller energy scale Δ (here Δ is the mean level spacing of the grain).

The addition spectrum manifests itself in transport experiments on electron tunneling between two leads through a quantum dot. The electrostatic potential of the dot, and therefore the energy of the electron addition E, can be controlled by a capacitively coupled gate electrode [3]. At temperatures T much smaller than E_c, the presence of the gap results in suppression of the conductance at almost all gate voltages. Only if the gate voltage is tuned to one of the discrete points of charge degeneracy, i.e., $E = 0$, is this suppression lifted. This phenomenon is known as the Coulomb blockade. The question arises whether and when the results of RMT (which does not take into account any charging gap) can be applied to the statistics of the conductance in the Coulomb blockade regime. Being interesting on its own, this question is also relevant for the recent experimental studies of the conductance fluctuations [4]. The answer to this question depends on how close the system is to the charge degeneracy point. Right at the degeneracy point, and at low temperatures $T \ll \Delta$, the transport occurs by means of resonant tunneling through a single state. Thus, the charging energy E_c is irrelevant, and RMT provides an adequate description of the distribution function of the peak heights [5,6]. Supersymmetry calculation [7] allows for a description of the crossover [8], with the magnetic field B, between the orthogonal ($B = 0$) and unitary ($B \to \infty$) ensembles.

The situation changes drastically if the system is tuned away from the charge degeneracy point, $E > \Delta \simeq T$. In this case the transport is due to the virtual transitions of an electron via excited states of the dot (so-called elastic cotunneling [9]); many levels with energies exceeding E contribute to the tunneling. The superposition of a large number of tunneling amplitudes changes the properties of the conductance fluctuations, which is studied for the first time in this Letter.

We will show that the correlation function $C(\Delta B)$ for the conductance in the cotunneling regime is universal (i.e., all the dependences for different E can be collapsed to a single curve upon rescaling of the magnetic field); magnetic correlation field B_c is inversely proportional to $\sqrt{E}$, and thus the charging energy controls the fluctuations of the elastic cotunneling. Furthermore, the functional form of $C(\Delta B)$ is entirely different from the known results [6]. Finally, we will find the distribution function of the conductance for all values of the magnetic field.

The quantum dot attached to two leads is described by the Hamiltonian

$$\hat{H} = \hat{H}_L + \hat{H}_R + \hat{H}_D + \hat{H}_T \,, \tag{1a}$$

where the Hamiltonians of the left (L) and right (R) leads are given by

$$\hat{H}_L = \sum_k \xi_k^l a_k^\dagger a_k, \qquad \hat{H}_R = \sum_k \xi_k^r b_k^\dagger b_k \,, \tag{1b}$$

and $\xi_k^{l,r}$ is the one-electron energy measured from the Fermi level. The Hamiltonian of the dot $\hat{H}_D$ has the form [3]

$$\hat{H}_D = \sum_k \xi_k c_k^\dagger c_k + E_c(\hat{n} - \mathcal{N})^2, \qquad \hat{n} = \sum_k c_k^\dagger c_k \,, \tag{1c}$$

where ξ_k describes the one-electron spectrum of the dot, and the second term in $\hat{H}_D$ corresponds to the charging energy, and $E_c = e^2/2C$. Here C is the capacitance of the dot, and $\mathcal{N}$ is the conventional dimensionless parameter related to the gate voltage V_g by $\mathcal{N} = V_g/eC_g$, with

C_g being the gate capacitance. The tunneling Hamiltonian couples the leads with the dot, and it has the form

$$\hat{H}_T = \sum_{k,p} t^l_{kp} a^\dagger_k c_p + \sum_{k,p} t^r_{kp} b^\dagger_k c_p + \text{H.c.} \tag{1d}$$

Operators a, b, and c in Eqs. (1b)–(1d) are the corresponding fermionic operators.

If tunneling is weak, the charge of the dot $\hat{n}$ is quantized. Obviously, degeneracy of the charging energy in Eq. (1c) corresponds to half-integer values $\mathcal{N}_m = m + \frac{1}{2}$ of the dimensionless gate voltage $\mathcal{N}$.

If V_g is tuned away from a degeneracy point, it takes a finite energy E to add one electron (or hole) to the dot,

$$E = E_c|\mathcal{N} - \mathcal{N}_m|, \qquad |\mathcal{N} - \mathcal{N}_m| < 1/2. \tag{2}$$

Positive (negative) values of $\mathcal{N} - \mathcal{N}_m$ correspond to the electronlike (holelike) lowest charged excitations.

We are considering the strong Coulomb blockade away from the resonance. Thus, we employ perturbation theory in the tunneling Hamiltonian (1d). The lowest nonvanishing contribution to the conductance G is

$$G = \frac{2\pi e^2}{\hbar} \sum_{k,p} |A_{kp}|^2 \delta(\xi^l_k)\delta(\xi^r_p). \tag{3}$$

The amplitude A_{kp} corresponds to the process in which an electron (hole) tunnels from state k in the left lead into a virtual state in the dot, and then it tunnels out to state p of the second lead. This amplitude is given by

$$A_{kp} = \sum_q t^l_{kq}(t^r_{pq})^* \frac{1}{|\xi_q| + E} \theta[\xi_q(\mathcal{N} - \mathcal{N}_m)]. \tag{4}$$

The denominator in Eq. (4) corresponds to the energy of virtual state q involved in the cotunneling process, and the step function $\theta(x)$ selects the dominating (electron or hole) channel.

In the most realistic case [4,10] of point contacts, Eqs. (3) and (4) may be further simplified. The tunneling matrix elements $|t^{l,r}_{kq}|^2$ do not depend on the indices k, q, and can be related to the conductances of the point contacts, $G_{l,r} = (2\pi e^2/\hbar)\nu_d \nu_{l,r}|t^{l,r}|^2$; here $\nu_{l,r,d}$ are the ensemble-averaged densities of states per area in the leads (l, r) and dot (d), respectively. Using these definitions, substituting Eq. (4) into Eq. (3), and performing the summation over k and p in Eq. (3), we find

$$G = \frac{\hbar}{2\pi e^2} G_l G_r |F(\mathbf{R}_l, \mathbf{R}_r)|^2. \tag{5}$$

The dimensionless function $F(\mathbf{R}_l, \mathbf{R}_r)$ contains all the information about elastic cotunneling through the dot between the point contacts located at $\mathbf{R}_l$ and $\mathbf{R}_r$,

$$F(\mathbf{R}_l, \mathbf{R}_r) = \frac{1}{\nu_d} \sum_q \frac{\psi^*_q(\mathbf{R}_l)\psi_q(\mathbf{R}_r)}{|\xi_q| + E} \theta[\xi_q(\mathcal{N} - \mathcal{N}_m)], \tag{6}$$

where ψ_q is the one-electron wave function in the closed dot. It is useful to rewrite F in terms of the retarded and advanced one-electron Green functions $\mathcal{G}^{R,A}$ of the dot,

$$F(\mathbf{R}_l, \mathbf{R}_r) = \frac{1}{\nu_d} \int \frac{d\omega}{2\pi i} \frac{\mathcal{G}^A_\omega - \mathcal{G}^R_\omega}{|\omega| + E} \theta[\omega(\mathcal{N} - \mathcal{N}_m)], \tag{7}$$

where

$$\mathcal{G}^{R,A}_\omega = \mathcal{G}^{R,A}_\omega(\mathbf{R}_l, \mathbf{R}_r) = \sum_q \frac{\psi^*_q(\mathbf{R}_r)\psi_q(\mathbf{R}_l)}{\omega - \xi_q \pm i0}.$$

(We put $\hbar = 1$ in all the intermediate calculations.)

Equations (5) and (7) express the elastic cotunneling conductance in terms of the exact electron wave functions of the dot. These functions vary strongly when the magnetic field is applied to the dot or the shape of the dot is changed. Thus, the conductance is a random quantity and one should consider different moments of the conductance distribution function. We will employ the ensemble averaging, which is equivalent to the averaging over applied magnetic field or over the peak index m. According to Eqs. (5) and (7), the averaged moments of the conductance are expressed in the terms of the averaged product of the Green functions. It is well known [11] that if the dot in the metallic regime (the transport mean free path or the size of the dot is much larger than the Fermi wavelength) and the relevant energies are much larger than Δ, these products can be related to the generalized classical correlators—diffuson $\mathcal{D}$ and Cooperon $\mathcal{C}$:

$$\langle \mathcal{G}^R_{\omega_1,B_1}(\mathbf{r},\mathbf{s})\mathcal{G}^A_{\omega_2,B_2}(\mathbf{s},\mathbf{r})\rangle = 2\pi\nu_d \mathcal{D}^{B_1,B_2}_{\omega_1-\omega_2}(\mathbf{r},\mathbf{s}), \tag{8a}$$

$$\langle \mathcal{G}^R_{\omega_1,B_1}(\mathbf{r},\mathbf{s})\mathcal{G}^A_{\omega_2,B_2}(\mathbf{r},\mathbf{s})\rangle = 2\pi\nu_d \mathcal{C}^{B_1,B_2}_{\omega_1-\omega_2}(\mathbf{r},\mathbf{s}), \tag{8b}$$

where $\langle\cdots\rangle$ stand for the ensemble averaging perfomed under the fixed magnetic fields B_1, B_2. The averages of the type $\langle G^R G^R\rangle$ and $\langle G^A G^A\rangle$ are much smaller and can be neglected. If the sample is dirty, so that the motion of electrons in the dot is diffusive, the diffuson and Cooperon (8) satisfy the equations

$$\left[-i\omega + D\left(-i\nabla_\mathbf{r} + \frac{e}{c}\mathbf{A}^-(\mathbf{r})\right)^2\right]\mathcal{D}^{B_1,B_2}_\omega = \delta(\mathbf{r} - \mathbf{s}), \tag{9a}$$

$$\left[-i\omega + D\left(-i\nabla_\mathbf{r} + \frac{e}{c}\mathbf{A}^+(\mathbf{r})\right)^2\right]\mathcal{C}^{B_1,B_2}_\omega = \delta(\mathbf{r} - \mathbf{s}), \tag{9b}$$

where D is the diffusion constant, and $\mathbf{A}^\pm$ is the vector potential due to the magnetic field, $\nabla \times \mathbf{A}^\pm = \mathbf{B}_1 \pm \mathbf{B}_2$. For the dot in the ballistic regime, the diffusion operator on the left-hand side of Eqs. (9) should be replaced with the Liouvillean operator. The solution of Eqs. (9) with the condition of vanishing normal component of the gauge invariant current at the boundary of the dot will enable us to find all the relevant correlation functions of the conductance and we are turning to this calculation now.

The averaged cotunneling conductance is obtained immediately by the averaging of Eq. (5) with the help of Eqs. (7) and (8a). The result is

$$\langle G\rangle = \frac{G_l G_r}{2\pi^2\nu_d e^2} \int_{-\infty}^{\infty} \frac{d\omega}{|\omega|} \mathcal{D}^{0,0}_\omega(\mathbf{R}_l, \mathbf{R}_r) \ln\frac{E + |\omega|}{E}. \tag{10}$$

Calculation of the second moment of the conductance is performed by taking into account that, because the relevant energy scale is much larger than Δ, the average of the product of four Green functions can be decoupled into a product of the pairwise averaged Green functions, which are in turn given by Eqs. (8). This yields

$$\langle G(B)G(B+\Delta B)\rangle = \left(\frac{G_l G_r}{2\pi^2 \nu_d e^2}\right)^2 \Bigg[\left| \int_{-\infty}^{\infty} \frac{d\omega}{|\omega|} \mathcal{D}_\omega^{0,0}(\mathbf{R}_l, \mathbf{R}_r) \ln \frac{E+|\omega|}{E} \right|^2 + \left| \int_{-\infty}^{\infty} \frac{d\omega}{|\omega|} \mathcal{D}_\omega^{B,B+\Delta B}(\mathbf{R}_l, \mathbf{R}_r) \ln \frac{E+|\omega|}{E} \right|^2 + \left| \int_{-\infty}^{\infty} \frac{d\omega}{|\omega|} C_\omega^{B,B+\Delta B}(\mathbf{R}_l, \mathbf{R}_r) \ln \frac{E+|\omega|}{E} \right|^2 \Bigg]. \tag{11}$$

The term in the second line of Eq. (11) corresponds to the square of the average conductance, and the last two terms describe the conductance fluctuations. The term in the third line of Eq. (11) depends only on ΔB [cf. Eq. (9a)], and it is present both for the orthogonal ($B = 0$) and for the unitary ($B \to \infty$) ensembles. To the contrary, the last term in Eq. (11) dies out for the unitary ensemble.

It is easily seen from Eq. (11) that the fluctuations are always of the order of the conductance itself. This may be understood from the following qualitative consideration. There are $N \sim E/\Delta \gg 1$ contributions corresponding to different eigenstates in the cotunneling amplitude (4). Assume that the phases of these contributions are completely random. Conductance is proportional to the modulus squared of the sum of these contributions, and thus there are N^2 terms in the conductance. Among those, N terms do not fluctuate, and the rest $N^2 - N$ are random. These random terms, however, do contribute to the fluctuation $\langle \delta G^2 \rangle$, and the number of nonvanishing terms in it is $N^2 - N$. Therefore, the average conductance is proportional to N, and its rms fluctuation is $\sim\sqrt{N^2 - N} \simeq N$. Thus, conductance in the cotunneling regime is not a self-averaging quantity despite a naive expectation that a large number of virtual states participating in the cotunneling may decrease the fluctuations.

Equations (10) and (11) are quite general; i.e., they are valid for an arbitrary relation between the energy of charged excitation E and Thouless energy $E_T \simeq \hbar D/L^2$ (here L is the linear size of the dot). In the most interesting regime, $E < E_T$, the correlation function of conductance fluctuations $C(\Delta B)$ acquires a universal form, as will be shown below.

Because $E < E_T$, the characteristic frequency ω in Eqs. (9) is much smaller than the lowest nonzero eigenvalue of the diffusion operator (which is of the order of D/L^2). Therefore, only the zero frequency mode can be retained in the solutions of Eqs. (9). This mode corresponds to the probability density homogeneously distributed over the dot, and the solution has the form

$$\mathcal{D}_\omega^{B_1,B_2} = \frac{S^{-1}}{-i\omega + \Omega_-}, \qquad C_\omega^{B_1,B_2} = \frac{S^{-1}}{-i\omega + \Omega_+}, \qquad \Omega_\pm = E_T \frac{S^2(B_1 \pm B_2)^2}{\Phi_0^2}, \tag{12}$$

where S is the area of the dot, $\Phi_0 = 2\pi\hbar c/e$ is the flux quantum, and the Thouless energy is given by $E_T = \alpha\hbar D/S$, with shape-dependent coefficient α of the order of unity. Equation (12) holds also for ballistic cavities; the only difference is that the expression for the Thouless energy changes to $E_T \simeq \hbar/\tau_{\mathrm{fl}}$, with τ_{fl} being the time of flight of an electron across the dot. Thouless energy can be independently measured by studying the correlation function of mesoscopic fluctuations for the same dot but with the contacts adjusted to the ballistic regime.

Substitution of Eq. (12) into Eq. (10) immediately yields the known [9] result for the averaged conductance

$$\langle G \rangle = \frac{\hbar G_l G_r}{2\pi e^2} \frac{\Delta}{E}. \tag{13}$$

However, the fluctuations $\delta G(B) = G(B) - \langle G \rangle$ are large. We find from Eq. (11) with the help of Eqs. (12),

$$\frac{\langle \delta G(B) \delta G(B+\Delta B)\rangle}{\langle G \rangle^2} = \Lambda\left(\frac{\Delta B}{B_c}\right) + \Lambda\left(\frac{2B+\Delta B}{B_c}\right), \tag{14}$$

where the scaling function $\Lambda(x)$ is given by

$$\Lambda(x) = \frac{1}{\pi^2 x^4}\left[\ln x^2 \ln(1+x^4) + \pi \arctan x^2 + \frac{1}{2}\mathrm{Li}_2(-x^4)\right]^2, \tag{15}$$

with $\mathrm{Li}_2(x)$ being the second polylogarithm function [12]. The asymptotic behavior of function Λ is $\Lambda(x) = 1 + (2x^2 \ln x^2)/\pi$, for $x \ll 1$ and $\Lambda(x) = (\pi x^2)^{-2} \ln^4 x^2$ for $x \gg 1$.

The correlation magnetic field B_c in Eq. (14) is controlled by the charging energy

$$B_c = \frac{\Phi_0}{S}\sqrt{\frac{E}{E_T}}. \tag{16}$$

It is worth noticing from Eqs. (13) and (16) that the correlation magnetic field B_c drops with approaching a charge degeneracy point (in agreement with the recent experiment [4]), whereas the quantity $\langle G \rangle B_c^2$ remains invariant. This invariance can be easily checked experimentally.

Let us present also the expression for the experimentally measurable correlation function of the conductance fluctuations $C(\Delta B) = \langle \delta G(B) \delta G(B+\Delta B)\rangle / \langle \delta G(B)^2 \rangle$. For both orthogonal and unitary ensembles we obtain from Eq. (14)

$$C(\Delta B) = \Lambda(\Delta B/B_c), \tag{17}$$

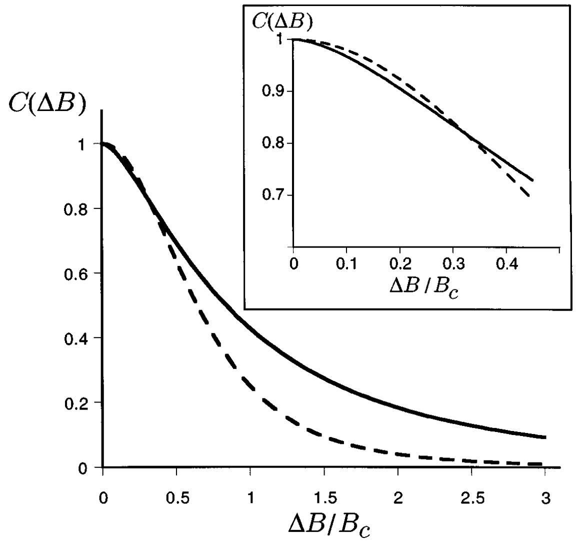

FIG. 1. The correlation function $C(\Delta B)$ for the conductance fluctuations in the elastic cotunneling regime (solid line) and for the peak height fluctuations $C = [1 + (\Delta B/B_c)^2]^{-2}$ (dashed line). For elastic cotunneling, $C(\Delta B)$ is nonanalytic at $\Delta B \to 0$; see inset, and Eq. (17).

function $\Lambda(x)$ is defined by Eq. (15). We emphasize that the functional form of $C(\Delta B)$ is different from the results for the peak heights fluctuations [6]; see Fig. 1.

As we saw, fluctuations of the conductance are of the order of the averaged conductance. Thus the distribution function is non-Gaussian; i.e., it is not characterized by its second moment only. Fortunately, for the elastic cotunneling the calculation of all the moments is possible which enables us to find the distribution function $P(g)$ [here we introduced random variable $g = G(B)/\langle G\rangle$]. The function $P(g)$ is defined as

$$P(g) \equiv \left\langle \delta\left(g - \frac{G}{\langle G\rangle}\right)\right\rangle = \int \frac{dq}{2\pi} e^{iqg} \sum_{n=0}^{\infty} \frac{(-iq)^n \langle G^n\rangle}{n!\,\langle G\rangle^n}. \tag{18}$$

Calculation of the average $\langle G^n(B)\rangle$ entering into Eq. (18) is performed in a fashion similar to the derivation of Eq. (11). In the universal regime $E < E_T$, we find

$$\frac{\langle G(B)^n\rangle}{\langle G\rangle^n} = \sum_{0\le j\le n/2} \frac{(n!)^2}{(j!)^2(n-2j)!}\left(\frac{\lambda}{4}\right)^j, \tag{19}$$

where we introduced the shorthand notation $\lambda \equiv \Lambda(B/B_c)$. The substitution of Eq. (19) into Eq. (18) yields

$$P(g) = \frac{\theta(g)}{\sqrt{1-\lambda}} \exp\left(-\frac{g}{1-\lambda}\right) I_0\left(\frac{g\sqrt{\lambda}}{1-\lambda}\right), \tag{20}$$

where $I_0(x)$ is the zeroth order modified Bessel function of the first kind. In the limiting cases $\lambda = 1$ and $\lambda = 0$, the distribution function $P(g)$ coincides with the Porter-Thomas distribution [1,13] for the orthogonal and unitary ensembles, respectively.

So far we considered elastic cotunneling only. It dominates over the inelastic processes [9] at $T < \sqrt{E\Delta}$, which is the typical regime for the modern experiments with semiconductor quantum dots [4,10]. At higher temperatures, the main conduction mechanism switches to the inelastic cotunneling. Nevertheless, the fluctuations are still determined by the elastic mechanism for $(E\Delta)^{1/2} \lesssim T \lesssim (E^2\Delta)^{1/3}$. At even higher temperatures, the inelastic contribution dominates also in the fluctuations. Their relative magnitude, however, is small, $\langle \delta G_{\rm in}^2\rangle/\langle G_{\rm in}\rangle^2 \simeq \Delta/T$. The correlation magnetic field is controlled by the temperature rather than by the charging energy and therefore is independent on the gate voltage.

In conclusion, we studied the statistics of mesoscopic fluctuations of the elastic cotunneling. We showed that the correlation magnetic field is controlled by the charging energy and the correlation function of the conductance is universal.

We are grateful to C.M. Marcus for illuminating discussions and reading the manuscript. This work was supported by NSF Grant No. DMR-9423244.

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