Stochastic Resonance with Sensitive Frequency Dependence in Globally Coupled Continuous Systems

Hu Gang,[1,2] H. Haken,[3] and Xie Fagen[4]
[1]Center of Theoretical Physics, CCAST (World Laboratory), Beijing 100080, China
[2]Department of Physics, Beijing Normal University, Beijing 100875, China*
[3]Institute of Theoretical Physics, University Stuttgart, Pfaffenwaldring 57/4, D-70550 Stuttgart, Germany
[4]Institute of Theoretical Physics, Academia Sinica, Beijing 100080, China
(Received 12 December 1995)

A model of globally coupled bistable continuous systems with two series of cells, modeling competing processes between activators and suppressors, subject to periodic and stochastic forces is investigated. In the large system size limit the exact solution of linear response theory is obtained, which shows typical stochastic resonance (SR) behavior. In particular, we find a resonance, sensitively depending not only on noise intensity and coupling strength, but also on the frequency of the periodic forcing. The SR occurs at a noise-induced Hopf bifurcation point, and is associated with the divergence of linear response. [S0031-9007(96)01039-3]

PACS numbers: 05.40.+j, 87.10.+e

The problem of stochastic resonance (SR) has attracted much attention since the last decade [1–8]. In recent years there has been a great interest to apply the SR concept to spatiotemporal systems [9–11]. In this direction, an extension of the model,

$$\dot{x} = ax - bx^3 + A\cos(\Omega t) + \Gamma(t), \quad a,b > 0, \quad (1)$$

which has been most extensively investigated in the SR study, to the coupled overdamped oscillators

$$\dot{x}_i = a_i x_i - b_i x_i^3 + \frac{\mu}{2m+1}\sum_{k=-m}^{k=m} x_{i+k} + A\cos(\Omega t) + \Gamma_i(t), \qquad i = 1,2,\ldots,L,$$
$$x_{i+L} = x_i, \qquad \langle\Gamma_i(t)\Gamma_j(t')\rangle = 2D\delta_{ij}\delta(t-t') \quad (2)$$

is most natural. Lindner *et al.* numerically studied Eqs. (2) in the nearest coupling case ($m = 1$) [10], and Jung *et al.* investigated a master equation describing globally coupled and two-state spin system [9], and Morillo *et al.* studied Eq. (2) with global coupling [11]. The physical significance of global coupling for practical systems, such as neural networks, multimode solid lasers and array of lasers, and Josephson junction systems has been well emphasized [9,11–13].

There has been a well known problem in the study of SR that one could not find active role played by frequency for the resonance, the system output is never peaked at certain nonzero frequency in the SR condition if it is plotted against the frequency of the forcing. The terminology stochastic resonance is not a resonance in the conventional sense; it refers only to the peaked behavior of the output with respect to the noise intensity. Recently, Gammaitoni *et al.* showed the relation between the resident time and the period of the external forcing [14]. Nevertheless the resonance behavior with respect to frequency in terms of the amplitude of the output signal has never been reported in the SR study.

In this Letter we will suggest a model with two kinds of cells indicated by $x_i, i = 1,2,\ldots,L$, and $y_j, j = 1,2,\ldots,N$ (without losing any generality we set $L = N$). The inner dynamics of each cell is described by Eq. (1), while all cells are globally coupled to each other through a single quantity $Z = \frac{\sum_{i=1}^{i=L} h_i x_i + \sum_{j=1}^{j=L} s_j y_j}{L}$. x cells are regarded active with positive h_i, while y cells are suppressive with negative s_j. Specifically we set $h_i = -s_j = 1$ for all i and j. The idea of the competition between the activators and suppressors appears in many fields. Then our model can be formulated as

$$\begin{aligned}
\dot{x}_i &= a_1 x_i - b_1 x_i^3 + \mu_1 Z(t) + A_1\cos(\Omega t + \gamma_1) + \Gamma_i(t)\\
\dot{y}_j &= a_2 y_j - b_2 y_j^3 + \mu_2 Z(t) + A_2\cos(\Omega t + \gamma_2) + \Delta_j(t)\\
\langle\Gamma_i(t)\rangle &= 0, \qquad \langle\Gamma_i(t)\Gamma_j(t')\rangle = 2D_1\delta_{ij}\delta(t-t')\\
\langle\Delta_j(t)\rangle &= 0, \qquad \langle\Delta_i(t)\Delta_j(t')\rangle = 2D_2\delta_{ij}\delta(t-t'), \qquad \langle\Gamma_i(t)\Delta_j(t)\rangle = 0,
\end{aligned} \quad (3)$$

where all parameters $a_{1,2}$, $b_{1,2}$ and $\mu_{1,2}$ are positive, and $Z(t) = X(t) - Y(t)$, $X(t) = \frac{\sum_{k=1}^{k=L} x_k}{L}$, $Y(t) = \frac{\sum_{k=1}^{k=L} y_k}{L}$. With this model we find a new type of resonance, which is purely noise induced, on one hand, and has sensitive dependence on frequency, on the other hand.

Now we start by writing down some known results of Eq. (1), which can be transformed to a Fokker-Planck equation (FPE)

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}[ax - bx^3 + A\cos(\Omega t)] \times P(x,t) + D\frac{\partial^2}{\partial x^2}P(x,t). \quad (4)$$

The asymptotic solution linearly responding the external forcing reads [6]

$$\langle x(t)\rangle = \mathrm{Re}\{AM\exp[i(\Omega t + \theta)]\} \quad (5)$$

$$\begin{aligned} M\exp(i\theta) &= \sum_{n=1}^{n=\infty} g_n \exp(i\alpha_n), \\ g_n &= \frac{1}{(\lambda_n^2 + \Omega^2)^{\frac{1}{2}}}\langle n|x|0\rangle\left\langle n \left| \frac{\partial}{\partial x}\right| 0\right\rangle \\ \cos(\alpha_n) &= \lambda_n/(\lambda_n^2 + \Omega^2)^{\frac{1}{2}}, \\ \sin(\alpha_n) &= -\Omega/(\lambda_n^2 + \Omega^2)^{\frac{1}{2}}, \end{aligned} \quad (6)$$

with $|n\rangle$ and $\langle n|$ being the nth right and left eigenvectors of the FP operator (4) with $A = 0$, respectively. The corresponding eigenvalues read $-\lambda_n$, which are ordered as $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$.

In Eqs. (2), with global coupling the summation of k runs from $k = 1$ to $k = L$, then all the cells are coupled to each other via the quantity $X(t) = \sum_{i=1}^{i=L} x_i(t)/L$. In the large system size limit the fluctuation of $X(t)$ must be negligibly small, then the motion of each cell can be computed through a single Langevin equation (LE), which can be transformed to a FPE

$$\frac{\partial P(x,t,X)}{\partial t} = -\frac{\partial}{\partial x}[ax - bx^3 + \mu X(t) + A\cos(\Omega t)] \times P(x,t,X) + D\frac{\partial^2}{\partial x^2}P(x,t,X) \quad (7)$$

(where we use identically $a_i = a, b_i = b, i = 1,2,\ldots,L$, and omit the subscript i since the equations for all cells are identical). The key point for solving (7) is that in the asymptotic state Eq. (7) can be closed in a self-consistent manner by setting

$$\langle x(t)\rangle = X(t) \quad (8)$$

where $\langle x(t)\rangle$ is the statistical average of x from Eq. (7). The identity (8) is due to the symmetry between different cells in the asymptotic state. Without the external forcing ($A = 0$), we can get the explicit stationary solution of Eq. (7) as $P(x,X) = N(X)\exp[-V(x,X)/D]$, $V(x,X) = -\frac{a}{2}x^2 + \frac{bx^4}{4} - \mu Xx$. Hence X can be solved from an implicit integral equation

$$\langle x\rangle = \int_{-\infty}^{\infty} xP(x,X)\,dx = X. \quad (9)$$

It is found that a pitchfork bifurcation of X (i.e., a spontaneous ordering phase transition) appears at a certain μ_c [12]. For $\mu < \mu_c$ the solution $X = 0$ is stable, while after $\mu > \mu_c$ this zero solution losses stability, and the two new nonzero stable solutions appear.

With small forcing ($A \ll 1$), we can separate $X(t)$ into two parts $X(t) = X_0 + X_t$, where $X_0 = \langle x\rangle_0$ is the stationary solution (9), and $X_t = \langle x\rangle_t = B\cos(\Omega t + \phi)$ is the oscillation part. Inserting $X(t)$ into Eq. (7), we can reduce the linear response problem to a simple algebraic equation of complex variable

$$B\exp[i(\Omega t + \phi)] = AM(X_0)\exp[i(\Omega t + \theta)] + \mu BM(X_0)\exp[i(\Omega t + \phi + \theta)], \quad (10)$$

leading to

$$X_t = \langle x\rangle_t = \mathrm{Re}\left\{\frac{AM(X_0)\exp[i(\Omega t + \theta)]}{1 - \mu M(X_0)\exp(i\theta)}\right\}, \quad (11)$$

where $M(X_0)$ and θ have exactly the same form as M and θ in (6).

The procedure for obtaining Eq. (11) can be directly applied to study the two-series systems (3), which correspond to the FPEs

$$\begin{aligned} \frac{\partial P(x,t)}{\partial t} &= -\left\{\frac{\partial}{\partial x}[a_1 x - b_1 x^3 + \mu_1 Z(t) + A_1\cos(\Omega t + \gamma_1)] + D_1\frac{\partial^2}{\partial x^2}\right\}P(x,t) \\ \frac{\partial P(y,t)}{\partial t} &= -\left\{\frac{\partial}{\partial y}[a_2 y - b_2 y^3 + \mu_2 Z(t) + A_2\cos(\Omega t + \gamma_2)] + D_2\frac{\partial^2}{\partial y^2}\right\}P(y,t). \end{aligned} \quad (12)$$

The spontaneous ordering phase transition can also appear in Eq. (12). In the following we are not interested in the phase transition of this kind, and will work in the region where stationary nonzero X and Y phases do not exist. With nonzero while very small $A_{1,2}$, we can assume $X(t) = B_1\cos(\Omega t + \phi_1)$, $Y(t) = B_2\cos(\Omega t + \phi_2)$. Inserting them into Eq. (12) we can immediately arrive at two coupled algebraic equations

$$B_{1,2}\exp(i\phi_{1,2}) = \mu_{1,2}M_{1,2}\exp(i\theta_{1,2})[B_1\exp(i\phi_1) - B_2\exp(i\phi_2)] + A_{1,2}M_{1,2}\exp[i(\theta_{1,2} + \gamma_{1,2})], \quad (13)$$

where the quantities $M_{1,2}$ and $\theta_{1,2}$ are given in Eq. (6). Equations (13) have solutions

$$B_{1,2}\exp(\phi_{1,2}) = \frac{Q_{1,2}}{Q_0}, \qquad Q_0 = 1 - \mu_1 M_1\exp(i\theta_1) + \mu_2 M_2\exp(i\theta_2),$$

$$Q_{1,2} = A_{1,2}M_{1,2}\exp[i(\gamma_{1,2} + \theta_{1,2})] \pm M_1 M_2\exp[i(\theta_1 + \theta_2)][A_{1,2}\mu_{2,1}\exp(i\gamma_{1,2}) - A_{2,1}\mu_{1,2}\exp(i\gamma_{2,1})]. \quad (14)$$

An obvious conclusion from the result (14) is that a great enhancement may take place as the denominator Q_0 vanishes. The parameter values for its vanishing can be determined by a set of two equations

$$\mu_1 M_1 \cos(\theta_1) = \mu_2 M_2 \cos(\theta_2) + 1,$$
$$\mu_1 M_1 \sin(\theta_1) = \mu_2 M_2 \sin(\theta_2). \quad (15)$$

It is emphasized again that Eqs. (14) and (15) are generally valid for arbitrary D, μ, and Ω, as well as for arbitrary forms of the potentials. A direct analysis on Eqs. (12) and on the theoretical results (14) will be presented in a forthcoming full paper. In the following we will study Eqs. (14) and (15) in the limits $\Omega \ll 1$, $\frac{a_{1,2}^2}{4b_{1,2}\{-\ln(\Omega)+\ln[-\ln(\Omega)]\}} < D_{1,2} \ll 1$, $\mu_{1,2} \ll 1$. Then we need to keep only the first term in (6), $M = g_1$, $\theta = \alpha_1$, and get

$$M_{1,2} = \frac{a_{1,2}\lambda_1(1,2)}{b_{1,2}D_{1,2}\sqrt{\lambda_1(1,2)^2 + \Omega^2}},$$
$$\cos(\theta_{1,2}) = \frac{\lambda_1(1,2)}{\sqrt{\lambda_1(1,2)^2 + \Omega^2}}, \quad (16)$$
$$\sin(\theta_{1,2}) = \frac{-\Omega}{\sqrt{\lambda_1(1,2)^2 + \Omega^2}},$$

where $\lambda_1(1,2)$ are the λ_1 eigenvalues computed from the two FPE's of (12), which can be made explicit as

$$\lambda_1(1,2) = \frac{\sqrt{2}a_{1,2}}{\pi b_{1,2}} \exp\left(-\frac{a_{1,2}^2}{4b_{1,2}D_{1,2}}\right). \quad (17)$$

Afterwards we simply denote $\lambda_1(1,2)$ by $\lambda_{1,2}$, respectively. Inserting (16) into (15), we get the condition for the divergence of the linear response

$$\lambda_1 + \lambda_2 = \frac{a_1\mu_1\lambda_1}{b_1D_1} - \frac{a_2\mu_2\lambda_2}{b_2D_2}, \quad (18)$$

$$\Omega^2 = \frac{(\lambda_1 - \lambda_2)(\frac{a_1\mu_1}{b_1}\lambda_1 + \frac{a_2\mu_2}{b_2}\lambda_2) - (D_1\lambda_1^2 + D_2\lambda_2^2)}{D_1 + D_2}. \quad (19)$$

In the case

$$\frac{a_1\mu_1}{b_1} = \frac{a_2\mu_2}{b_2} = \mu, \qquad D_1 = D_2 = D \quad (20)$$

we get rather simple and physically meaningful relations for the divergence

$$\mu = \frac{D(\lambda_1 + \lambda_2)}{\lambda_1 - \lambda_2}, \quad (21)$$

$$\Omega^2 = \lambda_1\lambda_2. \quad (22)$$

The condition (18) [or (21)] indicates the critical line μ_h for the possible divergence, while (19) [or (22)] specifies the actual frequency of divergence. This feature reminds us the behavior of Hopf bifurcation in deterministic systems. However, in the present case there is no trace of deterministic oscillation or deterministic Hopf bifurcation at or near μ_h. What we find is a purely noise-induced oscillation. In Figs. 1 and 2, we fix $a_2 = b_{1,2} = 1$, and $a_1 = 0.9$ and present some results for the case of (20). In Fig. 1(a) we plot the bifurcation curve μ_h vs D. Below this curve, the steady state $X = Y = 0$ is stable. At the curve, a divergence of linear response for small periodic perturbation occurs, the corresponding frequency of the divergent perturbation is shown in Fig. 1(b). Increasing μ over the curve μ_h, a spontaneous oscillation with the given frequency will take place. In Figs. 1(c) and 1(d) we give some numerical results of time-dependent evolution of the nonlinear system in the two-state approximation. The asymptotic spontaneous oscillation with vanishing external forcing [shown in (d)] is a noise-induced limit cycle. In Fig. 2(a) we set $A_1 = A_2 = A$ and fix $\gamma_1 = \gamma_2 = 0$, and plot $\eta_1 = \frac{B_1}{A}$ vs Ω at $\mu = 0.08$ for different D. A sensitive dependence of the amplification rate on the frequency is obvious, and divergence of the amplitude is observed when D approaches the critical point. In Fig. 2(b) we do the same thing for different μ at $D = 0.05$ and get similar results. In Figs. 2 the linear

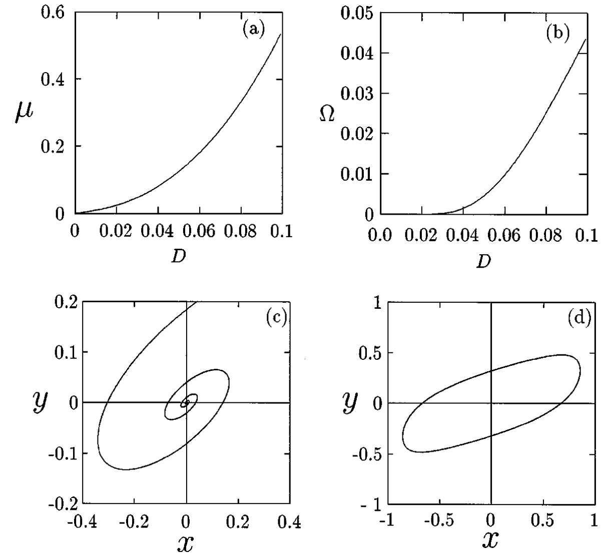

FIG. 1. $a_1 = 0.9$, $a_2 = b_1 = b_2 = 1$, $A_1 = A_2 = 0$. (a) μ_h vs D according to Eq.(21). (b) Ω vs D according to (22). (c) $D = 0.03$, $\mu = 0.04 < \mu_h$, the system approaches the steady state $X = Y = 0$ asymptotically. (d) $D = 0.03$, $\mu = 0.08 > \mu_h$. Spontaneous oscillation exists in the asymptotic state.

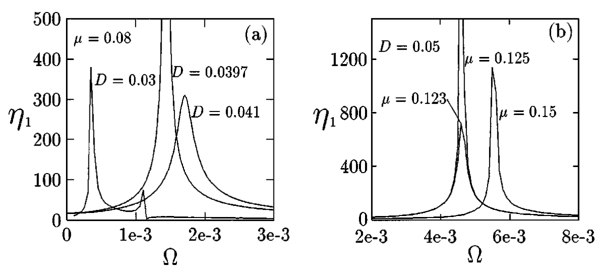

FIG. 2. $a_{1,2}$ and $b_{1,2}$ are the same as in Fig. 1. $A_1 = A_2$, $\gamma_1 = \gamma_2 = 0$. (a) $\eta_1 = \frac{B_1}{A}$ vs Ω for $\mu = 0.08$ and various D. (b) $D = 0.05$, η_1 vs Ω for various μ.

response in the region $\mu > \mu_h$ is computed with respect to the based spontaneous oscillation state, which is given numerically. It is interesting to notice the double-peak structure in Fig. 2(a). The second peak corresponds to the resonance with high-frequency harmonics. The behavior of $\eta_2 = \frac{B_2}{A}$ is similar to that in Figs. 2.

In conclusion, we have revealed a new kind of noise-induced oscillation. Many previous works talked about noise-induced oscillations, where coherent circulations exist in the deterministic dynamics either in the asymptotic state [15] or in the transient processes [16,17]. In our case, without noise there is no deterministic oscillation in both asymptotic state and transient process for the given parameters. The oscillation is purely noise induced. For the stochastic resonance, our model has some features common with the conventional SR (i.e., SR at zero frequency): The response curves are peaked with respect to D and μ due to the optimal modulation in double potential wells. The most interesting new feature is that the sensitive frequency dependence, which is the central point for conventional resonance problems while it is lacking in the conventional SR phenomena, can be clearly seen in our model. Then we can talk about a *general stochastic resonance* and enlarge the scope of SR study, including the conventional SR as the case of $\Omega = 0$. Moreover, we find divergence in the linear response solutions, which has never been found so far in the SR study. Three ingredients: noise, nonlinearity, and global coupling are crucial for the above features.

In this Letter we took $L \longrightarrow \infty$ for our analytic study. An estimation of the finite size effect is crucial for understanding the validity of the present theory in realistic systems. The key point in our analysis is to assume that the fluctuation of $Z(t) = X(t) - Y(t) = \frac{\sum_{n=1}^{L} x_i(t)}{L} - \frac{\sum_{n=1}^{L} y_i(t)}{L}$ can be neglected. For a theory of finite L, $Z(t)$ should be subject to certain fluctuations $Q(t)$ [$Z(t) = \langle Z(t)\rangle + Q(t)$ in (3)]. $Q(t)$ is of the order $L^{1/2}$ for a regular state and $L^{1/4}$ at μ_c and μ_h. Then all the above analysis works well even for finite system under the condition $\mu_{1,2}^2/L \ll D_{1,2}$ (or $\frac{\mu_{1,2}^2}{\sqrt{L}} \ll D_{1,2}$ at the second order bifurcations), only the divergences should be replaced by high while finite peaks.

This work is supported by the Foundation of National Natural Science of China and the Project of Nonlinear Science.

*Address for correspondence.

[1] R. Benzi, A. Sutera, and A. Vulpiani, J. Phys. A **14**, 453 (1981).
[2] C. Nicolis and G. Nicolis, Tellus **33**, 225 (1981).
[3] B. McNamara and K. Wiesenfeld, Phys. Rev. A **39**, 4854 (1989).
[4] L. Gammaitoni, F. Marchesoni, E. Menichella-Saetta, and S. Santucci, Phys. Rev. Lett. **62**, 349 (1989).
[5] T. Zhou and F. Moss, Phys. Rev. A **41**, 4255 (1990).
[6] G. Hu, G. Nicolis, and C. Nicolis, Phys. Rev. A **42**, 2030 (1990).
[7] P. Jung and P. Hanggi, Phys. Rev. A **44**, 8032 (1991).
[8] M. I. Dykman, R. Mannella, P. V. E. McClintock, and N. G. Stocks, Phys. Rev. Lett. **68**, 2985 (1992).
[9] P. Jung, U. Behn, E. Pantazelou, and F. Moss, Phys. Rev. A **46**, R1709 (1992).
[10] J. F. Lindner, B. K. Meadows, W. L. Ditto, M. E. Inchiosa, and A. R. Bulsara, Phys. Rev. Lett. **75**, 3 (1995).
[11] M. Morillo, J. Gomez-Ordonez, and J. M. Casado, Phys. Rev. E **52**, 316 (1995).
[12] M. Shiino, Phys. Rev. A **36**, 2393 (1987).
[13] K. Wiesenfeld, C. Bracikowski, G. James, and R. Roy, Phys. Rev. Lett. **65**, 1749 (1990).
[14] L. Gammaitoni, *et al.* (to be published).
[15] B. Gaveau, E. Gudovka-Nowak, R. Kapral, and M. Moreau, Phys. Rev. A **46**, 825 (1992); Physica (Amsterdam) **188A**, 443 (1992).
[16] P. Hanggi, P. Talkner, and M. Borkovec, Mod. Phys. Rev. **62**, 251 (1990).
[17] G. Hu, T. Ditzinger, C. Z. Ning, and H. Haken, Phys. Rev. Lett. **71**, 807 (1993).