# Experimental Determination of a Topological Invariant in a Pattern of Optical Singularities

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Topological invariants which control the organization and the evolution of ensembles of singularities may be evidenced in optical caustics. Our physical model is the optical pattern produced by light deflected through a nematic liquid crystal. We determine experimentally the Euler number of the critical set associated with the caustic and we identify the umbilics. We develop a method that enables us to determine the index of the umbilics and to distinguish between the two types of hyperbolic umbilics. We are then able to check, for the first time, the value of the topological invariant predicted by the recent Chekanov theory. [S0031-9007(96)00999-4]

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Fundamental laws of physics are often expressed with the help of invariant quantities: energy, linear momentum, angular momentum, etc. An invariant is a mathematical constraint that controls the evolution of the relevant physical quantities describing the state of a system. The concepts of topology have been widely applied in condensed matter or in quantum physics [1–4] and experimentally tested, for example, in the optics of glass fibers [5]. The introduction of topology in physics has widened the notion of invariant, and by topological invariant it is now understood some "object" which may be an integer, an algebraic structure, or a mathematical property, and which is preserved under any continuous modification of the system. Topological invariants are related, for instance, to discrete numbers such as topological charges, "winding numbers" like the Burger's vector of a point defect in a periodic pattern, etc. Thus they control, for instance, the spatial organization of singularities, and they also help give the rules that govern the transformations (or *bifurcations* or *metamorphoses*) of those ensembles of singularities. The typical example is given by the Poincaré-Hopf equality between the Euler number of a compact 2D surface and the total index of any vector field defined on it [6]. There the Euler number is a topological invariant. It means that the global organization of the singularities of a vector field is constrained by a condition depending only on the topological type of the surface.

The simplest concrete example showing ensembles of singularities is certainly provided by optical patterns [7]. Optical patterns (images) are obtained by sectioning, by a plane, the whole caustic produced by the rays emerging from a refracting medium or from a deflecting diopter. A caustic is the envelope of a set of rays and is generically composed of singular surfaces, lines, and points which form an ensemble of interconnected singularities. Whereas each singularity can be described locally by means of its normal form, the description of the ensemble of singularities needs here, too, the knowledge of the associated invariants. Caustics are, in fact, particular cases of singularities, the *Lagrangian* singularities, because of the Fermat principle [8]. In a 3D space, *Lagrangian* singularities have been classified into five types of decreasing dimension: the *folds* $A_2$ (surfaces), the *cusps* $A_3$ (lines), the *swallowtails* $A_4$, the *elliptic umbilics* $D_4^-$, and the *hyperbolic umbilics* $D_4^+$ which are point singularities [9]. The experiments on caustics have mostly concerned the local aspect of the singularities (their universal unfolding).

However, taking into account the eikonal equation results in a restriction of the definition of *Lagrangian* singularity into that of *optical* singularity [10]. The caustic is associated with the singularities of a mapping $\pi$ between two 3D spaces [8]. More precisely, the set of all the deflected rays constitutes a 3D ray manifold $R$ inside the phase space of greater dimension. The ray manifold is a regular surface without any singularity. By projecting it into the physical space $\mathbf{R}^3 = (x, y, z)$, one recovers the initial set of rays. The singular points are the points of $R$, where the rank of the projection $\pi$ is less than its maximal possible value: 3. These points form the critical (or singular) set $\Sigma \subset R$. This surface $\Sigma$ may itself possess singular *conical* points that are precisely the umbilics $D_4^\pm$. The caustic is the image $\pi(\Sigma) \subset \mathbf{R}^3$ of the critical set $\Sigma$. Chekanov [10] has defined a topological invariant as

$$I = \chi + 2D_4^\pm(-1/2), \tag{1}$$

involving the Euler number $\chi$ of the critical (compact) set and the number of umbilics $D_4^\pm$ of index $-1/2$. Among those umbilics of index $-1/2$ are the elliptic umbilics $D_4^-$. The hyperbolic ones have either an index $-1/2$ and they are named "*triangle*" ($D_4^{+t}$), or an index $+1/2$ and they are named "*drop*" ($D_4^{+d}$) [10,11] [see Figs. 3(a) and 3(b)]. It has been shown [10] that

$$\chi + 2(D_4^- + D_4^{+t}) = 0. \tag{2}$$

Here, for convenience, each symbol of type $D_4^\pm$ stands for the number of the respective singularities. To our knowledge the relation (2) has not yet been checked experimentally. In this Letter we shall adapt the theory to a concrete experimental case and give the first experimental evidence of the topological invariant (1).

  

The basic assumption in the theory that the critical set must be compact is hardly realized experimentally, because generally caustics contain infinite branches and also are limited by physical boundaries (area limited beams). But this strong constraint is circumvented by considering the points at infinity of the infinite branches. Each of these points represents a stationary ray. As a consequence, the caustic constitutes now a unique geometrical object comprising a real part and a virtual part connected together at infinity [7]. Secondly, if the optical pattern is periodic in two independent directions (biperiodicity), one may ignore any physical boundary in this plane, provided that the analysis is restricted to a unit cell of the pattern. The critical set $\Sigma$ is a double-sheeted [12] biperiodical surface. Each sheet, if taken apart, is topologically equivalent to a torus, and $\Sigma$ results from the linking of the two toruses by the *conical* singular points. It is then convenient to consider an optical system showing such a biperiodicity, for example, a light beam refracted on a biperiodic surface.

The experimental system used hereafter consists of a layer of nematic liquid crystal periodically distorted by the application of an electric field and through which light is transmitted [13]. The nematic liquid crystal is a uniaxial material in which the local optical axis is directed along the molecules and is denoted by the unit vector $\vec{n}$ (the director). For a well-defined value of the applied voltage, a biperiodic stationary structure named the "varicose" is developed [14]. In this structure the distortion of $\vec{n}$ is periodic in two directions of the $(x, y)$ plane of the nematic layer. The incident light beam is collimated and sent normally to the plane $(x, y)$. Inside the layer, the ordinarily polarized rays are not deviated and will not be considered hereafter. The extraordinary rays are deflected because along their trajectories the optical axis and thus the local refractive index varies. The caustic is the envelope of the set of these outgoing rays and of their prolongation backwards. The part of the caustic located above the layer is real, and the part located below is virtual but yet observable. The observation direction is along $z$. An image is the section of the caustic by the observation plane, for instance, here the focal plane of a microscope. By varying the height of this plane, one can observe the whole caustic surface, section after section.

In order to verify the formula (2), one first has to deduce the Euler number $\chi$ from the topology of the caustic, and also identify each of the umbilics. The appropriate way to determine the value of $\chi$ for a singular surface $\Sigma$ is to use a polygonal decomposition of it and to enumerate the number $v$ of vertices, $e$ of edges, and $f$ of faces [15]. The Euler number is defined as $\chi = v - e + f$. Note that each conical point $D_4^{\pm}$ (i.e., each umbilic) in $\Sigma$ must be associated with one vertex. The $\chi$ of each torus deconnected from its twin is zero [15]. In the reconnection process of two half-cones that give a $D_4^{\pm}$, the two vertices reduce to one (see Fig. 1). Then the final $\chi$ is the sum of the initial ones (i.e., zero) minus the number $D_4^{\pm}$ of umbilics. In our case we count 8 umbilics by unit cell, therefore, $\chi = -8$. Another method would be to define on the critical set $\Sigma$ a Morse function and to enumerate its extrema: $m$ being the number of its minima, $M$ of its maxima, and $S$ of its saddle points. The Euler number would be in the case of a regular surface the sum $\zeta = m - S + M$ [1,6]. This method is not, in principle, valid for singular surfaces, but it may give additional information on the topology as we shall see in the discussion.

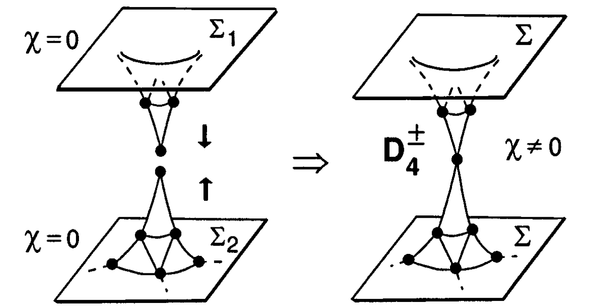


FIG. 1. The critical set $\Sigma$ is a surface composed of two sheets, each one of Euler number $\chi = 0$ (each sheet $\Sigma_{1,2}$ is, in fact, a torus represented here as a rectangle with periodic boundaries). The surfaces are tiled with curvilinear triangles (see text). The full $\Sigma$ is obtained by reconnection and now $\chi$ is $\neq 0$ by elimination of one vertex at the conical point $D_4^{\pm}$.

Now we enumerate the different types of umbilics. In a plane section, the fold surfaces appear as curved lines and the cusp lines as the tips of semicubical parabolas. The identification of the punctual singularities, i.e., swallowtails, elliptic, and hyperbolic umbilics, is made using sections at several heights (Fig. 2). As the height is increased, an elliptic umbilic appears as a small curved triangle, which reduces itself to a single point at the singularity. A hyperbolic umbilic appears as a cusp tip located inside a curve and forming a corner at the singularity. We find per unit cell a number $D_4^- = 4$ of elliptic umbilics, and a number $D_4^+ = 4$ of hyperbolic umbilics. Now there remains to distinguish between the two types of hyperbolic umbilics. There is actually no technique to directly determine experimentally the index of a given hyperbolic umbilic. This problem is solved here by calculating the different geometrical elements [10,11] which define the index, using a model that is able to reproduce exactly the experimental caustic (the details of the calculation shall be presented in a forthcoming publication).

The whole caustic is calculated using for the director field $\vec{n} = (\cos\varphi\cos\psi, \cos\varphi\sin\psi, \sin\varphi)$ in the biperiodic varicose structure, a form which is deduced from the experimental findings $\varphi = a\{\sin(px + qy) + b\sin(p'x + q'y)\}\sin rz$, $\psi = 0$, where $r = \pi/d$, $d$ is the thickness of the nematic layer, and $(p, q)$ and $(p', q')$ are the wave vectors defining the biperiodic structure. This form is found to reproduce exactly the experimental caustic with $a = 0.6$, $b = 0.5$, $(p, q) = (0.0115\ \mu\text{m}^{-1}, -0.0048\ \mu\text{m}^{-1})$, $(p', q') = (0.0115\ \mu\text{m}^{-1}, 0.0048\ \mu\text{m}^{-1})$, $d = 308\ \mu\text{m}$.

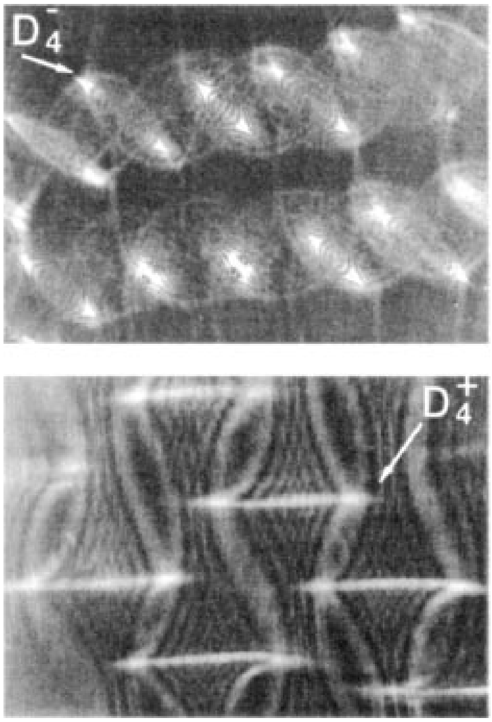


FIG. 2. Sections by the focal plane, of the experimental caustic produced by a biperiodic pattern. At two different heights one evidences the elliptic umbilics $D_4^-$ (top), and the hyperbolic umbilics $D_4^+$ (bottom).

The differential equations for the rays inside the nematic layer are the 3D generalization of those we had obtained in the 2D case [7,16]. Numerically, we model the incident beam by a set of $5 \times 10^4$ parallel rays. Each incoming ray is parametrized by two numbers $\lambda$ and $\mu$, which are the coordinates of the intersection of the ray by a plane normal to it. The outgoing rays are characterized by four parameters: their position $(x_0, y_0)$ in the upper interface and their direction $\alpha, \beta$. From the functional dependence of these parameters on $\lambda$ and $\mu$, we determine the location of each type of singularity $A_2$, $A_3$, $A_4$, and $D_4^{\pm}$, by the method of Thom's classes [12,17]. We check that the calculated ensemble of singularities coincides with the one observed in the experiment: four real $D_4^-$, four virtual $D_4^+$, and 12 $A_4$ per unit cell. Now we represent the singular set in the ray manifold which is here parametrized by $\lambda$, $\mu$, and $z$. In this system of coordinates, the kernel $P$ of the Lagrangian projection is the $(\lambda, \mu)$ plane, and the characteristic direction $d$ is along $z$. We find that for each of the hyperbolic umbilics, where the singular set is locally a cone, the kernel separates the characteristic direction $d$ from the $A_3$ line (see Fig. 3). Hence, all the hyperbolic umbilics present in the experimental caustic are of index $+1/2$, i.e., they are of the *drop* type $D_4^{+d}$. Therefore there is no hyperbolic umbilic with index $-1/2$, i.e., of type *triangle* in the experiment: $D_4^{+t} = 0$.

We have thus found that each unit cell of the optical pattern is characterized by a Euler number $\chi = -8$ and contains four elliptic umbilics $D_4^-$, zero hyperbolic umbilic of type *triangle* $D_4^{+t}$, and four hyperbolic umbilics of type *drop* $D_4^{+d}$. Hence, the sum $-8 + 2(4 + 0) = 0$, and the Chekanov formula is experimentally verified. The other method valid only for measuring the $\chi$ of regular surfaces may be used now to characterize further the topology. There one must define a generic function $f$ on $\Sigma$. If we choose, for instance, $f = 1/(z - z_0)$, experimentally the function level $f = C$ will correspond to the section of the caustic by the focal plane of the microscope. The plane $z = z_0$ separates the real from the virtual part of the caustic. We may show that there this method gives the same $\chi$ *under the condition that there is no triangle hyperbolic umbilic* $D_4^{+t}$. More precisely, one finds that the sum $\zeta = m - S + M$ is equal to $-2D_4^-$, so that $\zeta = \chi + 2D_4^{+t}$. In one unit cell of our optical pattern we count two minima, four maxima, and 14 saddle points. Finally, one checks that $\zeta = 2 - 14 + 4 = -8 = \chi$.

The Chekanov formula may be used to characterize any ensemble of optical singularities and its transformations. In general, there remains, however, the practical problem of distinguishing between the hyperbolic umbilics. There are two particular cases where the solution is obvious: when the number of $D_4^{+t}$ is equal either to

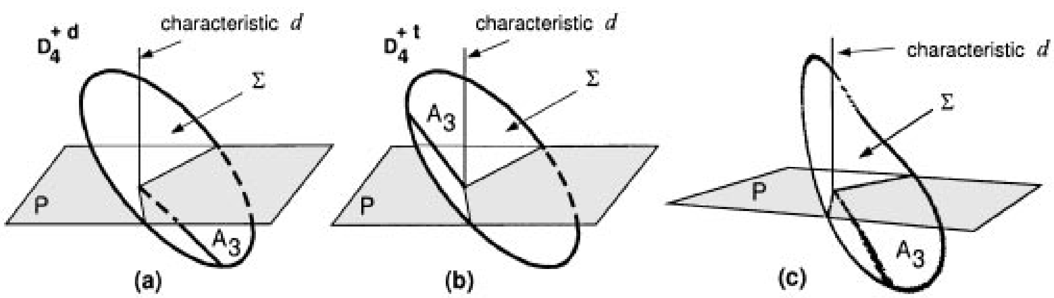


FIG. 3. In the neighborhood of a $D_4^+$ point, the critical set $\Sigma$ is a cone. The kernel $P$ of the Lagrangian projection at the point $D_4^+$ cuts the cone. If it separates the characteristic $d$ from the line $A_3$, the hyperbolic umbilic is of the "*drop*" type (a). It is of the "*triangle*" type in the other case (b). See Ref. [11]. The calculation from the physical model, showing (c) the critical set $\Sigma$, the cusp line $A_3$, the kernel $P$, and the characteristic $d$ going through a hyperbolic umbilic identical to the experimental one. This umbilic is obviously of the *drop* type (a).

zero, or to the number of observed hyperbolic umbilics. In general, the formula is unable to determine the index of each hyperbolic umbilic. If one associates the result of our measurement $\chi = -D_4^{\pm}$ with the Chekanov relation $\chi + 2(D_4^- + D_4^{+t}) = 0$, one deduces that $D_4^- + D_4^{+t} - D_4^{+d} = 0$ which simply expresses the conservation of the total index (here equal to zero). As a consequence, we may now define exactly the type of umbilics involved in the Zakalyukin transformations [18]:

$$D_4^- + 2A_4 \rightleftharpoons D_4^{+t}, \tag{3}$$

$$D_4^{+t} + D_4^{+d} \rightleftharpoons \varnothing . \tag{4}$$

Note that similar relations were obtained, but in the frame of regular wave surfaces [19]. The annihilation or creation of one pair of hyperbolic umbilics [relation (4)] is the only transformation which can modify the topology of $\Sigma$ by changing the $\chi$ value by $\pm 2$. Other types of invariants exist that may prove also useful to characterize ensembles of optical singularities such as the cobordism invariants [20]. There a result is that the number of swallowtails $A_4$ and of umbilics $D_4^{\pm}$ must be even [21]. This is checked in our experiment, since we observe 12 $A_4$ and eight $D_4^{\pm}$ per unit cell. The physical conditions, as well as the symmetries, might also impose additional topological constraints. Moreover, the experimental caustics are the envelope of *straight* rays which are emitted by a *punctual* source. These features bring in a restriction which is not considered in the actual theory, and they might induce new effects, for instance, they might allow new additional metamorphoses or forbid some.

In conclusion, we have experimentally studied an ensemble of optical singularities and determined its Chekanov invariant. The condition of compacity of the critical set is fulfilled by choosing a unit cell in a biperiodic pattern. The difficult problem of distinguishing the different types of hyperbolic umbilics is solved by numerically reconstructing from a very realistic model the local topology of these singularities. It is, to our knowledge, the first example of an experimental evidence of a topological invariant associated with a system of rays in geometrical optics. Our results confirm the Chekanov theory, and they show that the full description of ensembles of singularities and of their transformations thus becomes more accessible to the experimentalist. More generally, the possibility of a direct determination of topological invariants provides new powerful tools for the experimental study of the singularities encountered in the various fields of physics (condensed matter, wave propagation, shocks, hydrodynamics, etc.).

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