# Intermittent Loss of Synchronization in Coupled Chaotic Oscillators: Toward a New Criterion for High-Quality Synchronization

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We observe incomplete synchronization of coupled chaotic oscillators over a wide range of coupling strengths and coupling schemes for which high-quality synchronization is expected. Long intervals of high-quality synchronization are interrupted at irregular times by large, brief desynchronization events that can be attributed to "attractor bubbling," clearly demonstrating that the standard synchronization criterion is not always useful in experiments. We suggest a simple method for rapidly selecting the coupling schemes that are most likely to produce high-quality synchronization. [S0031-9007(96)01028-9]

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Spontaneous synchronization of dynamical systems, such as that appearing in clocks [1] and fireflies [2], for example, has been the subject of curiosity and scholarly inquiry for many years. Recently, several research groups [3] have synchronized *chaotic* systems; a surprising result considering that initially close trajectories of chaotic systems diverge exponentially. One motivation for researching chaos synchronization techniques is to explore their practical application to various problems in communication [4], optics [5], and nonlinear dynamics model verification [6]. Also, a detailed understanding of the synchronization process may lead to new schemes for controlling complex spatiotemporal dynamics that occur during cardiac fibrillation [7] or in diode laser arrays [8], for example.

Recent reports indicate that our understanding of the synchronization process is not complete: two well-matched chaotic systems do not necessarily synchronize under conditions when high-quality synchronization is expected [9–12]. Rather, long intervals of high-quality synchronization are interrupted irregularly by large (comparable to the size of the attractor), brief desynchronization events that may be undesirable or even harmful in some applications. It is proposed [9–12] that this behavior, called *attractor bubbling*, is associated with invariant sets embedded within the synchronization manifold that are unstable to perturbations caused by noise or slight parameter mismatch [13].

The primary objectives of this Letter are to demonstrate that the popular and widely used criterion for synchronization of coupled chaotic oscillators entirely fails to predict the regime of high-quality (burst-free) synchronization in a simple experimental system, and to compare our observations with recent theories [9–12]. A secondary objective is to suggest a new, simple method for estimating the range of high-quality synchronization.

In our investigation, we consider one-way coupling of two chaotic electrical circuits. The dynamical evolution of a single circuit [14], shown schematically in Fig. 1, is governed by the set of dimensionless equations

$$\dot{V}_{1j} = \frac{V_{1j}}{R_1} - g[V_{1j} - V_{2j}], \tag{1a}$$

$$\dot{V}_{2j} = g[V_{1j} - V_{2j}] - I_j, \qquad j = m, s\,, \tag{1b}$$

$$\dot{I}_j = V_{2j} - R_4 I_j\,, \tag{1c}$$

where $V_{1j}$ and $V_{2j}$ represent the voltage drop across the capacitors (normalized to the diode voltage $V_d = 0.58$ V), $I_j$ represents the current flowing through the inductor (normalized to $I_d = V_d/R = 0.25$ mA for $R = \sqrt{L/C} = 2,345\ \Omega$), $g[V] = V/R_2 + I_r[\exp(\alpha V) - \exp(-\alpha V)]$ represents the current (normalized to $I_d$) flowing through the parallel combination of the resistor and diodes, and time is normalized to $\tau = \sqrt{LC} = 2.35 \times 10^{-5}$ sec. The circuit displays "double scroll" behavior for $I_r = 2.25 \times 10^{-5}$, $\alpha = 11.6$, $R_1 = 1.2$, $R_2 = 3.44$, $R_3 = 0.043$, $R_{\mathrm{dc}} = 0.15$ (the dc resistance of the inductor), and $R_4 = R_3 + R_{\mathrm{dc}} = 0.193$, where all resistances have been normalized to $R$.

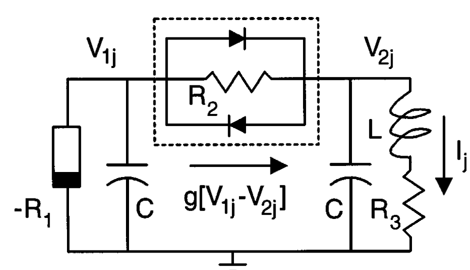


FIG. 1. Chaotic electronic oscillator consisting of a negative resistor $R_1 = 2814\Omega$, capacitors $C = 10$ nF, an inductor $L = 55$ mH (dc resistance $353\Omega$), a resistor $R_3 = 100\Omega$, and a passive nonlinear element (resistor $R_2 = 8,067\Omega$, diodes type 1N914, dashed box).

The dynamics of the coupled system can be expressed succinctly as

$$\dot{\mathbf{x}}_m = \mathbf{F}[\mathbf{x}_m], \tag{2a}$$

$$\dot{\mathbf{x}}_s = \mathbf{F}[\mathbf{x}_s] - c\mathbf{K}(\mathbf{x}_m - \mathbf{x}_s), \tag{2b}$$

where $\mathbf{x}_m$ ($\mathbf{x}_s$) denotes the position in $n$-dimensional phase space of the master (slave) oscillators, $\mathbf{F}$ represents the flow of the oscillators, $\mathbf{K}$ is an $n \times n$ coupling matrix, $c$ is the scalar coupling strength, and $\mathbf{x}_j^T = (V_{1j}, V_{2j}, I_j)$. We match all components to within 1%, construct ten circuits, and select two from the group whose bifurcation diagrams are most similar (there are only slight differences in the bifurcation diagrams for all ten). To facilitate our discussion of the synchronization process, we introduce new coordinates $\mathbf{x}_\parallel = (\mathbf{x}_m + \mathbf{x}_s)/2$ and $\mathbf{x}_\perp = (\mathbf{x}_m - \mathbf{x}_s)/2$ that specify the dynamics within and transverse to the synchronization manifold, respectively.

Synchronization of the oscillators occurs when $\mathbf{x}_s(t) = \mathbf{x}_m(t) \equiv \mathbf{s}(t)$ which is equivalent to $\mathbf{x}_\perp(t) = 0$; the system resides on an $n$-dimensional synchronization manifold within the $2n$-dimensional space. In practice, the occurrence of high-quality synchronization is indicated by $|\mathbf{x}_\perp(t)| < \varepsilon$, where $\varepsilon$ is a length scale small (typically 1%) in comparison to the typical dimension of the chaotic attractor. We note that the synchronization condition has been generalized [15] to include the possibility that the variables of the slave oscillator are equal to a function of the variables of the master oscillator.

The widely used criterion for synchronization of coupled chaotic oscillators was proposed by Fujisaka and Yamada [16] over a decade ago. They investigate the stability of the synchronized state $\mathbf{x}_\perp = 0$ by determining the transverse Liapunov exponents $\lambda_\perp^1 \geq \lambda_\perp^2 \geq \cdots \geq \lambda_\perp^n$ characterizing the dynamics transverse to the synchronization manifold. The exponents are determined from the solution to the variational equation

$$\delta\dot{\mathbf{x}}_\perp = \{D\mathbf{F}[\mathbf{s}(t)] - c\mathbf{K}\}\delta\mathbf{x}_\perp, \tag{3}$$

obtained by linearizing Eq. (2) about $\mathbf{x}_\perp = 0$, where $D\mathbf{F}[\mathbf{s}(t)]$ denotes the Jacobian of $\mathbf{F}$ evaluated on $\mathbf{s}(t)$. They propose that high-quality synchronization occurs for values of the coupling strength $c$ where $\lambda_\perp^1 < 0$.

In stark contrast to the expected results, we observe incomplete synchronization for all coupling schemes over a wide range of coupling strengths where $\lambda_\perp^1 < 0$. For example, consider "$V_2$ coupling" ($K_{22} = 1$, $K_{ij} = 0$ otherwise) of the oscillators for $c = 4.6$. A numerical analysis of Eq. (3) reveals that $\lambda_\perp^1 < 0$ when $c > c_{\text{crit}}^{22} \simeq 0.64$. As seen in Fig. 2, we observe brief, large-scale intermittent desynchronization events in the experimentally observed temporal evolution of $|\mathbf{x}_\perp(t)|$. *This behavior persists indefinitely.*

To demonstrate that the standard synchronization criterion fails entirely for the $V_2$-coupled oscillators, we measure the average distance from the synchronization manifold $|\mathbf{x}_\perp(t)|_{\text{rms}}$, which is sensitive to the global

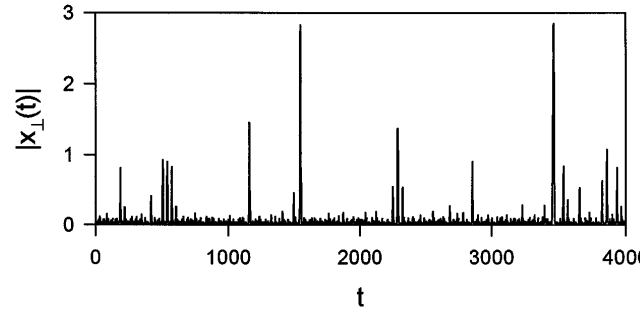


FIG. 2. Experimentally observed intermittent loss of synchronization in $V_2$-coupled chaotic oscillators for $c = 4.6$. Long intervals of high-quality synchronization [$|x_\perp(t)| \approx 0$] are interrupted by brief, large-scale (comparable to the size of the synchronization manifold) desynchronization events. The characteristic time scale of the system corresponds to $\sim 6\tau = 0.141$ msec.

transverse stability of the synchronized state [17], and the maximum observed value of the distance from the manifold $|\mathbf{x}_\perp(t)|_{\max}$, which is sensitive to the local stability of the state [10]. Figure 3(a) shows the experimentally measured values of $|\mathbf{x}_\perp(t)|_{\text{rms}}$ (solid line) and $|\mathbf{x}_\perp(t)|_{\max}$ (dashed line) as a function of the coupling strength. Compare these measurements to the predicted values of $\lambda_\perp^1$ (solid line) in Fig. 3(b). It is seen that $|\mathbf{x}_\perp(t)|_{\text{rms}}$ decreases rapidly as the coupling strength increases and that

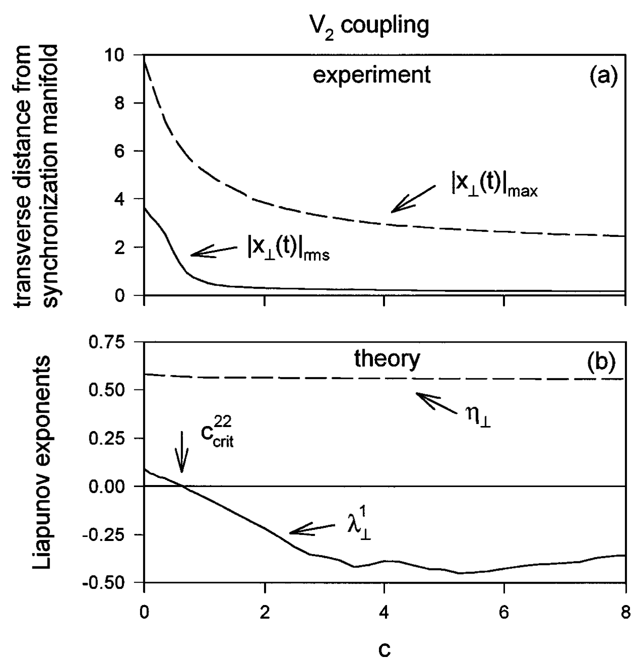


FIG. 3. (a) Experimentally observed degree of synchronization and (b) theoretically predicted stability of the synchronized state for $V_2$-coupled chaotic oscillators. We observe desynchronization events for $c > c_{\text{crit}}^{22}$, as indicated by $|\mathbf{x}_\perp(t)|_{\max} \gg |\mathbf{x}_\perp(t)|_{\text{rms}} \approx 0$. High-quality synchronization is never observed for this coupling scheme even though the standard synchronization criterion predicts its occurrence for $\lambda_\perp^1 < 0$. Recent theories predict attractor bubbling for $\eta_\perp > 0$, in agreement with our observations.

it is near zero for $c > c_{\text{crit}}^{22}$, where $\lambda_\perp^1 < 0$. This observation indicates that our model of the electrical circuit accurately describes its dynamics [17]. Persistent desynchronization events occur for $c > c_{\text{crit}}^{22}$, where $|\mathbf{x}_\perp(t)|_{\max}$ remains large; high-quality, bubble-free synchronization [$|\mathbf{x}_\perp(t)|_{\max}$ comparable to the noise level] is never observed. Surprisingly, similar results are found for most other coupling schemes.

We find that high-quality synchronization can only be obtained for coupling schemes where $K_{11} \neq 0$, although the range is less than that expected based on the standard synchronization criterion. For example, consider "$V_1$-coupled" oscillators ($K_{11} = 1$, $K_{ij} = 0$ otherwise) where high-quality synchronization is expected for $c > c_{\text{crit}}^{11} \simeq 0.305$ based on a numerical analysis of Eq. (3). Figure 4(a) shows the experimentally observed variation of $|\mathbf{x}_\perp(t)|_{\text{rms}}$ (solid line) and $|\mathbf{x}_\perp(t)|_{\max}$ (dashed line) with a coupling strength which should be compared to the predicted values of $\lambda_\perp^1$ (solid line) shown in Fig. 4(b). Again, it is seen that $|\mathbf{x}_\perp(t)|_{\text{rms}}$ decreases rapidly as the coupling strength increases and that it is near zero for $c > c_{\text{crit}}^{11}$ where $\lambda_\perp^1 < 0$. Note that high-quality synchronization is only obtained for coupling strengths much greater than expected.

Our results indicate that the criterion proposed by Fujisaka and Yamada [16], and widely used in theoretical studies of synchronization [3,4,6], is not a sufficient condition for high-quality synchronization of chaotic oscillators

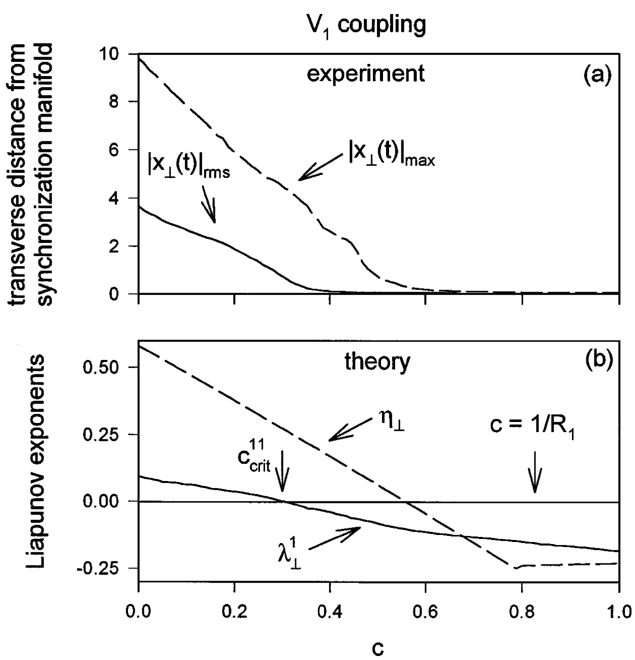


FIG. 4. (a) Experimentally observed degree of synchronization and (b) theoretically predicted stability of the synchronized state for $V_1$-coupled chaotic oscillators. High-quality synchronization [$|\mathbf{x}_\perp(t)|_{\max} \approx 0$] is observed only for $\eta_\perp < 0$. The range of high-quality synchronization predicted by our approximate method is $c > 1/R_1$.

in an experimental setting. Recent research [9–12] suggests that the criterion fails when the transverse Liapunov exponents characterizing invariant sets embedded within the synchronization manifold are greater than zero under conditions when $\lambda_\perp^1 < 0$. A desynchronization event corresponds to the growth of a perturbation (due to noise or parameter variation) during the interval when the trajectory is in a neighborhood of these invariant sets.

To test this hypothesis, we determine the most transversely unstable invariant set, characterized by its maximum transverse Liapunov exponent $\eta_\perp$, since it mediates the transition from attractor bubbling to high-quality synchronization. A numerical analysis of the low-period unstable orbits [18] indicates that the unstable steady-state $\mathbf{x}_\parallel = 0$ is the most unstable set. Figures 3(b) and 4(b) show the dependence of $\eta_\perp$ on the coupling strength (dashed line) for $V_2$- and $V_1$-coupled oscillators, respectively. For $V_2$-coupled oscillators, it is seen that $\eta_\perp > 0$ for all $c$, consistent with our observation that desynchronization events occur for all $c$. For $V_1$-coupled oscillators, it is seen that the transition to high-quality synchronization ($|\mathbf{x}_\perp|_{\max}$ comparable to the noise level) occurs near the point where $\eta_\perp$ becomes less than zero. The transition is not sharp, which may be the result of the finite noise level and parameter variation in the experiment.

Based on our observations, it appears that the proper criterion for high-quality synchronization of chaotic oscillators is $\eta_\perp < 0$. While this criterion is mathematically precise, it may be difficult to apply in practice because there are an infinite number of invariant sets whose stability must be determined [18]. Is there a different method for estimating the range of high-quality synchronization that captures the essence of the mathematically precise criterion without being overly complex? We believe that recent studies of the dynamics of linear systems characterized by non-normal matrices offers some guidance. Trefethen [19] shows that perturbations can grow significantly in the transient phase of the dynamics of a linear system even when the eigenvalues of the matrix governing the dynamics are all negative and distinct. Hence, the eigenvalues do not necessarily say much about the behavior of the system in the transient phase, rather they characterize the asymptotic, long-term behavior.

In a similar vein, our observations suggest that the Liapunov exponents characterizing the dynamics of a nonlinear system do not necessarily say much about the transient behavior. Noise and the unstable invariant sets give rise to persistent transient behavior in which the effects of perturbations are magnified during brief intervals. A simple method for testing whether perturbations can grow in the transient phase is to investigate the time derivative of the Liapunov function $\mathcal{L} = |\delta\mathbf{x}_\perp(t)|^2$ computed from a mathematical model of the system. The function $\mathcal{L}$ is equal to the square of the distance between the trajectory and the synchronized state $\mathbf{x}_\perp = 0$ for small distances [20]. A sufficient condition that all perturbations decay to the manifold without transient growth is

$$\frac{d\mathcal{L}}{dt} = 2\delta \mathbf{x}_{\perp}(t) \cdot \{(D\mathbf{F}[\mathbf{s}(t)] - c\mathbf{K})\delta \mathbf{x}_{\perp}(t)\} < 0 \quad (4)$$

for all times. We suggest that condition (4) can be used to quickly estimate the range of coupling strengths that result in high-quality, burst-free synchronization of coupled chaotic oscillators. Note that the Liapunov function depends on the choice of the metric (it is not invariant), and hence it underestimates the range of high-quality synchronization.

For the $V_2$-coupled oscillators, $d\mathcal{L}/dt = 2\{R_1^{-1}\delta x_1^2 - g'[V_{1m}(t) - V_{2m}(t)](\delta x_1 - \delta x_2)^2 - c\delta x_2^2 - R_4 \delta x_3^2\}$, where $\delta \mathbf{x}_{\perp}(t) = (\delta x_1, \delta x_2, \delta x_3)$, and $g'[V] = R_2^{-1} + \alpha I_r[\exp(\alpha V) + \exp(-\alpha V)]$. We see that $d\mathcal{L}/dt$ can be greater than zero regardless of the value of the coupling strength $c$. Hence, attractor bubbling may be present for all $c$, in agreement with our experimental observations. For the $V_1$-coupled oscillators, $d\mathcal{L}/dt = 2\{(R_1^{-1} - c)\delta x_1^2 - g'[V_{1m}(t) - V_{2m}(t)](\delta x_1 - \delta x_2)^2 - R_4 \delta x_3^2\}$ can be greater than zero for all times when $c < R_1^{-1} = 0.83$, also in reasonable agreement with our observations. These results suggest that our method is useful for estimating the range of high-quality synchronization without the need of complex numerical calculations.

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