Collective Excitations of Atomic Bose-Einstein Condensates

Mark Edwards*
Department of Physics, Georgia Southern University, Statesboro, Georgia 30460-8031

P. A. Ruprecht and K. Burnett*
Clarendon Laboratory, Department of Physics, University of Oxford, Parks Road, Oxford OX1 3PU, United Kingdom

R. J. Dodd† and Charles W. Clark
Electron and Optical Physics Division, National Institute of Standards and Technology, Technology Administration,
U. S. Department of Commerce, Gaithersburg, Maryland 20899
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We apply linear-response analysis of the Gross-Pitaevskii equation to obtain the excitation frequencies of a Bose-Einstein condensate confined in a time-averaged orbiting potential trap. Our calculated values are in excellent agreement with those observed in a recent experiment. [S0031-9007(96)00981-7]

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The recent attainment of quantum degeneracy conditions in magnetically trapped alkali vapors [1–3] has opened the road to understanding the many-body physics of atomic Bose-Einstein condensates (BECs) in unprecedented detail. For dilute gases, it is believed that the essential physics of the BEC ground state is captured in the Gross-Pitaevskii mean-field formalism. Calculations done with the Gross-Pitaevskii (GP) equation [4,5] have indeed agreed reasonably well with the few experimental determinations of condensate shapes, sizes, and lifetimes that have been made to date, but it cannot be said that the theory has been subject to stringent tests.

In this paper we report theoretical results for the excitation spectrum of a BEC of trapped ^{87}Rb, obtained by computing the response to small mechanical disturbances of a BEC described by the GP equation. These results are compared with those of a recent experiment [6], which has observed the free oscillations of a BEC that is briefly shaken at frequencies near resonance. We believe that this comparison provides the most critical quantitative test of mean-field theory made to date. The agreement between experimental and theoretical results is excellent.

To describe a magnetically trapped atomic gas, we adopt the standard GP equation, which is applicable [7–9] when the condensate fraction of a gaseous system is close to unity. Each atom in the condensate occupies the same orbital $\psi_g(\mathbf{r})$, which is determined by solution of the nonlinear Schrödinger equation,

$$[H_0 + N_0 U_0 |\psi_g(\mathbf{r})|^2]\psi_g(\mathbf{r}) = \mu \psi_g(\mathbf{r}), \qquad (1)$$

where $H_0 = -\frac{\hbar^2}{2m}\nabla^2 + V_{\mathrm{trap}}(\mathbf{r})$ is the Hamiltonian for an isolated atom in the trap, N_0 is the number of atoms in the condensate, U_0 represents the interaction between condensate atoms, and the eigenvalue μ is the chemical potential.

In most current trap designs, V_{trap} can be described by the anisotropic harmonic oscillator potential, $V_{\mathrm{trap}}(\mathbf{r}) = \frac{1}{2} m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$, where m is the atomic mass and $\omega_i = 2\pi\nu_i$ is the angular frequency of oscillation along the axis i. The time-averaged orbiting potential (TOP) trap [1,10] treated here is cylindrically symmetric; its potential is given by $\omega_x = \omega_y = \omega_\perp$ and $\omega_z = \sqrt{8}\,\omega_\perp$. The results presented below correspond to $\nu_z = \omega_z/2\pi = 210$ Hz. The parameter U_0 expresses the interaction between two atoms as $U_0 = 4\pi\hbar^2 a/m$, where a is the scattering length, which characterizes the zero-energy behavior of the s-wave phase shift in collisions between two atoms. The scattering length a is the only piece of atomic collision data used as input to our calculations. The present results are given in terms of the most recent [11] experimental value, $a_e = 110 a_0$, where a_0 is the Bohr radius; our calculations were actually carried out with a previously published [12] value of $a_t = 100 a_0$. Since Eq. (1) obeys a scaling law involving N_0, $\nu_\perp$, and a (see below), we can rescale our results to compare quantitatively with experiment. We have also performed the calculation for the exact conditions of the experiment of Ref. [6]. It should be noted that for the alkali atoms in current BEC studies, the experimental determination of a requires extensive spectroscopic analysis. Present values of a are accompanied by substantial uncertainties [13] for some of the alkalis; this is not, however, the case for ^{87}Rb.

Equation (1) has previously been solved by several independent methods [4,5] to describe the BEC ground state. Here we investigate the response of the ground state to an oscillatory perturbation at angular frequency ω_p using standard linear-response theory [14]. The associated time-dependent GP equation takes the form

$$i\hbar\frac{\partial\Psi}{\partial t} = [H_0 + U_0|\Psi(\mathbf{r},t)|^2 + f_+(\mathbf{r})e^{-i\omega_p t} + f_-(\mathbf{r})e^{i\omega_p t}]\Psi(\mathbf{r},t), \qquad (2)$$

where $f_\pm(\mathbf{r})$ are the spatially dependent amplitudes of the perturbation. We solve this equation in the linear-response limit. The details of this approach are described elsewhere [15], and we simply state the central results here. By using the form

$$\Psi(\mathbf{r},t) = e^{-i\mu t/\hbar}[N_0^{1/2}\psi_g(\mathbf{r}) + u(\mathbf{r})e^{-i\omega_p t} + v^*(\mathbf{r})e^{i\omega_p t}] \quad (3)$$

we obtain Eq. (1) and also the linear-response equations,

$$[\mathcal{L} - \hbar\omega_p]u(\mathbf{r}) + N_0U_0[\psi_g(\mathbf{r})]^2 v(\mathbf{r}) = - N_0^{1/2} f_+(\mathbf{r}) \times \psi_g(\mathbf{r}), \quad (4)$$

$$N_0U_0[\psi_g^*(\mathbf{r})]^2 u(\mathbf{r}) + [\mathcal{L} + \hbar\omega_p]v(\mathbf{r}) = - N_0^{1/2} f_-^*(\mathbf{r}) \times \psi_g^*(\mathbf{r}), \quad (5)$$

where $\mathcal{L} = H_0 - \mu + 2U_0N_0|\psi_g(\mathbf{r})|^2$.

This pair of equations can be solved by expansion in terms of the GP normal-mode equations,

$$[\mathcal{L} - \hbar\omega_\lambda]u_\lambda(\mathbf{r}) + N_0U_0[\psi_g(\mathbf{r})]^2 v_\lambda(\mathbf{r}) = 0, \quad (6)$$

$$N_0U_0[\psi_g^*(\mathbf{r})]^2 u_\lambda(\mathbf{r}) + [\mathcal{L} + \hbar\omega_\lambda]v_\lambda(\mathbf{r}) = 0, \quad (7)$$

where ω_λ is an eigenvalue and $u_\lambda(\mathbf{r})$, $v_\lambda(\mathbf{r})$ are corresponding eigenfunctions. In solving the above equations we demand that $u_\lambda(\mathbf{r})$ and $v_\lambda(\mathbf{r})$ be square integrable and, in solving Eq. (1), we require that

$$\int d^3r|\psi_g(\mathbf{r})|^2 = 1. \quad (8)$$

The ground-state condensate wave function $\psi_g(\mathbf{r})$ for the TOP trap is symmetric under rotations about the z axis; it follows that Eqs. (6) and (7) are invariant under such rotations, so the eigenfunctions $u_\lambda(\mathbf{r})$ and $v_\lambda(\mathbf{r})$ are characterized by sharp eigenvalues m of the z projection of the angular momentum.

As we have discussed elsewhere [15], there is a straightforward connection between the resonant oscillation frequencies ω_λ and the quasiparticle mode frequencies that are encountered. Stated simply, Eqs. (6) and (7) are identical to the equations that define the quasiparticle modes and frequencies within the Bogoliubov approximation. Thus an experiment that measures the free oscillatory response of a shaken BEC provides a direct observation of the quasiparticle spectrum. In particular, by shaking the BEC at a frequency near one of the resonances ω_λ, one can produce a response that is dominated by that ω_λ. This is the approach that has been taken by the first such experiment on the system [6].

We have solved numerically the system of equations consisting of Eqs. (1), (6), and (7) under the conditions of that experiment. The solution was accomplished in two steps. First, Eq. (1) was solved by expanding the solution $\psi_g(\mathbf{r})$ in a basis set consisting of a finite number of trap eigenfunctions. The details of the numerical method have been recounted elsewhere [5]. Equations (6) and (7) were then solved by expanding $u_\lambda(\mathbf{r})$ and $v_\lambda(\mathbf{r})$ in the same basis set. These expansions convert Eqs. (6) and (7) into a generalized matrix eigenvalue problem that can be solved by standard numerical techniques [15]. The error in the solutions was assessed by increasing the basis-set size until the mode frequencies converged to at least three figures.

Figure 1 shows our results for the lowest three excitation frequencies (in units of the trap frequency $\nu_\perp^{(t)}$), as a function of N_0. A simple scaling law facilitates comparison of calculation and experiment. A solution of Eqs. (1), (6), and (7), plus normalization for experimental values of the parameters $\{N_0^{(e)}, a_e, \nu_\perp^{(e)}, m_e\}$, will also satisfy the equations for the parameter set $\{N_0^{(t)}, a_t, \nu_\perp^{(t)}, m_t\}$, if the quantity $\gamma = N_0 a(m\nu_\perp)^{1/2}$ is constant. Thus, excitation measurements performed on a BEC with $N_0^{(e)}$ atoms in a trap of frequency $\nu_\perp^{(e)}$ can be related to the spectrum displayed in Fig. 1 by taking

$$N_0^{(t)} = \left(\frac{a_e}{a_t}\right)\left(\frac{\nu_\perp^{(e)}}{\nu_\perp^{(t)}}\right)^{1/2} N_0^{(e)}, \quad (9)$$

where $\nu_\perp^{(t)} = 210/\sqrt{8} \approx 74.25$ Hz.

We have used this scaling law to compare our results with those of the recent experiment [6], where excitations of an atomic BEC were observed for the first time. Included on this graph are the data points of Ref. [6] rescaled using Eq. (9) to match the trap, scattering-length, and condensate-number parameters of our calculation ($\omega_\perp = 210/\sqrt{8}$ Hz, $a_t = 100a_0$). As a check of the scaling law, we have also performed the calculation for the precise parameters of the experiment ($\nu_\perp = 43.2$ Hz, 132 Hz, and $a = 110a_0$). Table I presents the comparison. As expected, the results are identical to those predicted by using the scaling law and the previous calculation. No attempt has been made here to account for experimental uncertainties. The ranges presented for the experimental

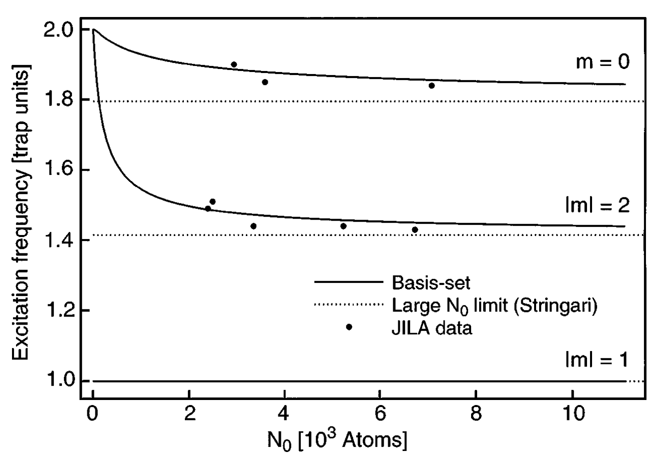

FIG. 1. The lowest three calculated excitation frequencies, in units of the perpendicular trap frequency $\nu_\perp^{(t)}$, of the JILA TOP-trap condensate as a function of the number of condensate atoms, $N_0^{(t)}$. The dotted lines show the large-N_0 predictions of Ref. [17], and the filled circles show the raw data points of Ref. [6] rescaled according to Eq. (9).

TABLE I. A comparison of the data of Ref. [6] with a calculation performed using the actual experimental trap, scattering-length, and condensate-number parameters. Note that the first column of the table indicates the azimuthal quantum number of the mode.

m	$\nu_\perp^{(e)}$	$N_0^{(e)}$	$\frac{\nu_{\text{mode}}^{(e)}}{\nu_\perp^{(e)}}$	$\nu_\perp^{(t)}$	$N_0^{(t)}$	$\frac{\nu_{\text{mode}}^{(t)}}{\nu_\perp^{(t)}}$	% diff.
0	43.2	3420	1.84–1.88	43.2	3421	1.89	2.7
0	132	2400	1.79–1.83	132	2401	1.88	5.0
2	43.2	2800	1.41–1.44	43.2	2801	1.49	5.7
2	132	2200	1.39–1.42	132	2203	1.47	5.8

ratio ($\nu_{\text{mode}}^{(e)}/\nu_\perp^{(e)}$) in Table I involve a (1–3)% reduction of the raw data to extrapolate to zero-amplitude driving [16]. It is fair to say that the agreement is excellent, as the difference between theory and experiment ranges from 2% to 6%.

To better understand the nature of these excitations we compare them with the results of Stringari [17] who obtained analytic solutions to the linearized GP equation in the hydrodynamic limit ($N_0 \to \infty$). The curve labeled "$|m| = 1$" is a doubly degenerate dipole excitation that coincides exactly with the first excited state of the bare trap. This is because the lowest dipole mode of an ensemble of identical interacting atoms in an external harmonic potential corresponds to a rigid motion of the center of mass, independent of the nature of interatomic forces [17].

The curve labeled "$|m| = 2$" corresponds to the two degenerate excitations that have magnetic quantum numbers $m = \pm 2$. The excitation frequencies tend to 2ω in the noninteracting ($N \to 0$) limit, as expected for a two-dimensional harmonic oscillator. The quadrupolar nature of these excitations is exhibited in Fig. 2, which contains a plot of u_λ over a region of the x-y plane [i.e., $u_\lambda(x, y, 0)$] for $N_0 = 2000$ atoms. This plot clearly shows the four-peak structure characteristic of quadrupole excitations, which is simply the angular dependence $\cos(2\phi)$. The large-N_0 limit of this mode, $\nu_{\text{mode}}/\nu_\perp \to \sqrt{2}$, is also shown in Fig. 1. The fact that the middle curves also seem to be approaching this limit as $N_0 \to \infty$ also lends support to their interpretation as quadrupole excitations.

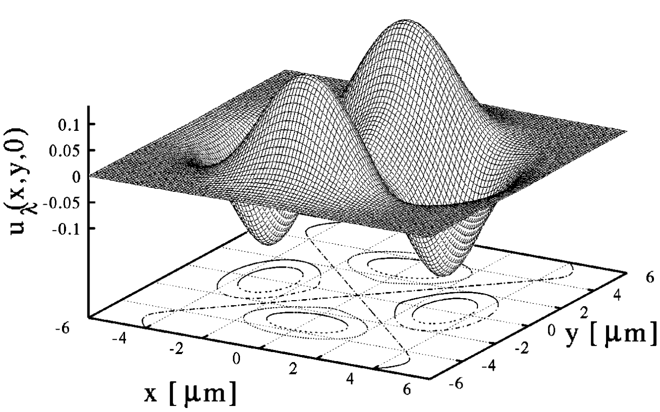

FIG. 2. The u_λ component of the $|m| = 2$ excitation (middle curve of Fig. 1) displaying its quadrupolar shape.

The curve labeled "$m = 0$" corresponds to a "large-N_0" excitation that is a breathing mode in the x-y plane. The asymptotic limit is shown in Fig. 1 and is given by [17]

$$\nu_{\text{mode}}/\nu_\perp \to \left(2 + \frac{3}{2}\lambda^2 - \frac{1}{2}\sqrt{9\lambda^4 - 16\lambda^2 + 16}\right)^{1/2}, \tag{10}$$

where $\lambda = \nu_z/\nu_\perp = \sqrt{8}$ for the TOP trap. In the noninteracting limit, the frequencies of the breathing and quadrupole modes are seen to degenerate, reflecting a well-known property of the two-dimensional harmonic oscillator. As discussed in Ref. [6], the symmetries of the normal modes can be tested by experimental selection rules, and the classifications of the observed modes are found to agree with those given here.

In conclusion, we have presented excitation spectra that agree well with the data of a recent experiment. We have shown that these data constitute a direct measurement of the $T = 0$ Bogoliubov spectrum of an atomic BEC. We have also used the large-N_0 limit and mode shapes to describe the nature of these excitations. The experimental confirmation of these data will have significant implications for understanding the many-body physics of these dilute, weakly interacting bosonic systems, and for practical use in future BEC engineering.

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*Also at Physics Laboratory, National Institute of Standards and Technology, Technology Administration, U.S. Department of Commerce, Gaithersburg, MD 20899.
†Also at Institute for Physical Science and Technology, University of Maryland at College Park, College Park, MD 20742.

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