Level Statistics and Localization for Two Interacting Particles in a Random Potential

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We describe the level statistics of two interacting particles in a disordered conductor by a Gaussian matrix ensemble with preferential basis. This simplified model allows us to identify the energy scale E_U, below which the spectrum exhibits the universal Wigner-Dyson rigidity. $E_U \propto |U|$, when a small interaction $|U|$ yields "Rabi oscillations" between only two eigenstates at $U = 0$, while $E_U \propto U^2$ for larger $|U|$, as numerically confirmed for conductors. Scaling to insulators, this yields a localization length increasing as $|U|$ before eventually behaving as U^2. [S0031-9007(96)00896-4]

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For a single particle diffusing in a disordered system of size L smaller than the one particle localization length L_1 there are two characteristic energies: the Thouless energy $E_c = \hbar D/L^2$ and the level spacing $\Delta_1 \approx B_1/L^d$ (B_1, D, and d are the bandwidth, the diffusion constant, and the system dimension, respectively). If one writes the distribution of energy levels as a Gibbs factor of a fictitious Coulomb gas, the corresponding pairwise interaction for levels with separation $\epsilon < E_c$ coincides [1] with the logarithmic repulsion characteristic of the matrix ensembles which are statistically invariant under change of basis, e.g., the Gaussian orthogonal ensemble (GOE). For $\epsilon > E_c$, the level repulsion vanishes more or less quickly, depending on the system dimension. The dimensionless conductance g_1 is given by E_c/Δ_1. This ratio is the single relevant parameter in the scaling theory of localization. In quasi-one-dimension, the size where $g_1 \approx 1$ defines L_1. In three dimensions, the mobility edge is characterized by $g_1 \approx g_c$, where g_c is of order 1.

We shall generalize those concepts to two particles with a local (repulsive or attractive) interaction. This two interacting particle (TIP) problem has received particular attention since Shepelyansky [2] pointed out that certain TIP states may extend over a scale L_2 much larger than L_1. Shepelyansky's original reasoning consists in mapping the problem for $L \gg L_1$ onto a random band matrix model with a superimposed diagonal matrix. Imry [3] later used the Thouless scaling block picture to arrive at precisely the same results. The smearing due to the interaction of the energy levels within L_1 was estimated using Fermi's golden rule, yielding $L_2 \propto U^2$. This delocalization effect has been confirmed by transfer matrix studies [4,5], and unambiguously illustrated from numerical studies [6] of rings threaded by an AB flux. However, in one dimension, for system sizes which can be numerically investigated, one obtains [5] $L_2 \propto |U|$ contrary to Fermi's golden rule, and a disorder dependence [4] $L_2 \propto L_1^{\alpha}$ with $\alpha \approx 1.5-1.7$ and not 2, as predicted by Shepelyansky and Imry.

To understand those contradictory results, we study the TIP energy level statistics at a scale L_1 in order to identify the energy which plays the role of E_c in this case, and to determine its dependence on U. We assume a tight-binding model [2] on a d-dimensional lattice (L_1^d sites p, random potential with a box distribution of width $2W$). The hopping term takes a constant value $V = 1$ and U is the on-site interaction. Assuming two electrons with opposite spins, we consider the symmetric states. The TIP Hamiltonian [2] can be written in a basis of the $N = L_1^d(L_1^d + 1)/2$ (symmetrized) products of one particle states $|AB\rangle$. We denote by R_{pA} the value on site p of the one particle eigenstate with energy ϵ_A. On this basis, the diagonal terms are dominated by one particle contributions $\epsilon_A + \epsilon_B$ and the interaction Hamiltonian yields a full matrix (for $L \leq L_1$) with entries $UQ_{ABA'B'} = U\sum_p R_{pA}^* R_{pB}^* R_{pA'} R_{pB'}$. The magnitude of those terms is of order $U/L_1^{3d/2}$ with a random sign.

Before considering the TIP Hamiltonian, it is instructive to discuss a simplified matrix model where the correlations between matrix elements are neglected: an ensemble of real symmetric matrices G with independent entries, characterized by Gaussian distributions with variances $\langle G_{ii}^2\rangle \approx B_1^2/3$ ($B_1 = 4Vd + 2W$) and $\langle G_{ij}^2\rangle \approx U^2/L_1^{3d}$ for the diagonal and off-diagonal terms, respectively. The averages are set to zero, which neglects a shift of the diagonal terms by an amount $UQ_{ABAB} \approx U/L_1^d$ assumed to be much smaller than B_1. These shifts preserve the sign of U and, for large U, eventually split the energy band into two parts. For the sake of simplicity, we ignore them, restricting us to small U and to a Gaussian matrix with preferential basis (GMPB) model which has been used previously [7] to study the GOE to Poisson crossover for the level statistics, and to define a maximum entropy model [8], where the range of the level interaction depends on a parameter. When $\langle G_{ii}^2\rangle$ is very large as compared to $\langle G_{ij}^2\rangle$, one has indeed a strongly preferential basis and it is convenient to reorder the diagonal terms such that $G_{11} < G_{22} < \cdots < G_{NN}$. Those G_{ii} may be considered as the positions of the energy levels in the zeroth approximation, when the interaction with the other states is neglected. The small coupling terms G_{ij} spread those basis states over Γ/Δ_2 neighbors. $\Delta_2 \approx \sqrt{2\pi\langle G_{ii}^2\rangle}/N$ is the level spacing and Γ can be estimated

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using Fermi's golden rule: $\Gamma \approx 2\pi\langle G_{ij}^2\rangle/\Delta_2$. Diagonalizing G by an orthogonal transformation O ($G = OG_dO^t$, where G_d is a diagonal matrix with real entries E_α), we consider the Wigner strength function $\rho_W(E,n) = \sum_{\alpha=1}^N O_{n\alpha}^2\delta(E - E_\alpha)$. For $L \gg L_1$, the ensemble average of $\rho_W(E + G_{nn}, n)$ has been found [9–11] in agreement with the Breit-Wigner form

$$\langle \rho_W(G_{nn} + E, n)\rangle = \frac{\Gamma}{2\pi[E^2 + \Gamma^2/4]} \equiv \mathcal{R}_\Gamma(E). \quad (1)$$

We show in Fig. 1 that this form characterizes also the original TIP Hamiltonian for $L \leq L_1$, once the shifts UQ_{nn} of the quasienergies H_{nn} are taken into account. For the squared amplitude $O_{n\alpha}^2$ of the projection of an eigenvector $|\alpha\rangle$ on a basis state $|n\rangle$ separated in energy by $E = E_\alpha - G_{nn}$, this implies [12] that its ensemble average is of order $\Delta_2\mathcal{R}_\Gamma(E)$. This shows us that a basis state $|n\rangle \equiv |AB\rangle$ (i.e., an eigenstate of the TIP Hamiltonian for $U = 0$) becomes delocalized over Γ/Δ_2 of its neighbors (i.e., over the basis states $|n'\rangle$ where $H_{n'n'}$ is close to $H_{nn} = \epsilon_A + \epsilon_B$), with a Lorentzian shape centered in $H_{nn} + UQ_{nn}$. Γ determines the localization in the preferential basis, and is given by Fermi's golden rule for small U ($U \leq 2$ in Fig. 1).

Having understood how the eigenstates are delocalized by the interaction over the preferential basis, we focus our attention on the energy levels. We introduce a symmetry breaking parameter μ in the probability density

$$\rho(G) \propto \exp\left(-\sum_{i=1}^N \frac{G_{ii}^2}{2\sigma^2} - (1 + \mu)\sum_{i<j}^N \frac{G_{ij}^2}{\sigma^2}\right), \quad (2)$$

with $\sigma^2 \approx B_1^2/3$ and $\sigma^2/2(1 + \mu) \approx U^2/L_1^{3d}$. When $\mu = 0$, one recovers the GOE ensemble with $\rho_{\mathrm{GOE}}(G) \propto \exp[-tr(G^2)/2\sigma^2]$. When $\mu \neq 0$, there is a factor

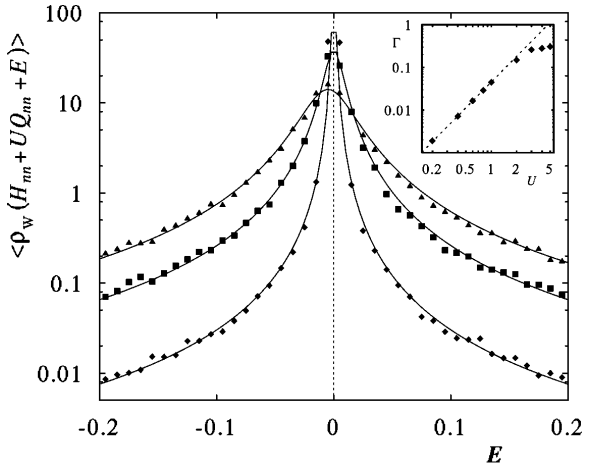

FIG. 1. The ensemble averaged strength function for a TIP Hamiltonian (5 by 5 lattice in the metallic regime $W = 2, V = 1$). Diamonds, squares, and triangles are for $U = 0.2$, 0.6, and 1.0, respectively. Lines are Breit-Wigner functions $\mathcal{R}_\Gamma(E)$, fitted to the numerical data. The inset shows how Γ depends on U. The line represents $\Gamma = U^2/22$.

$\rho_\mu(G)$ which removes the statistical invariance under change of basis. Expressed [8] in eigenvalue-eigenvector coordinates, it reads

$$\rho_\mu(G) \propto \prod_{\alpha<\beta}^N \exp\left(-\frac{\mu}{2\sigma^2}(E_\alpha - E_\beta)^2\sum_p O_{p\alpha}^2 O_{p\beta}^2\right). \quad (3)$$

The question is to understand how this additional factor, after integration over the matrices O [distributed with Haar's measure $\mu(dO)$ over the orthogonal group], can destroy the logarithmic level repulsion coming from the measure $\mu(dG) = \prod_{\alpha<\beta}^N |E_\alpha - E_\beta| \prod_\alpha^N dE_\alpha\,\mu(dO)$. This will allow us to identify the characteristic scale E_μ below which one recovers the GOE rigidity, and above which the levels become uncorrelated. Two cases have to be considered.

(i) $\Gamma < \Delta_2$.—The G_{ij} are so small that one can just consider the coupling between two nearest neighbor diagonal entries, i.e., a 2×2 matrix which can be diagonalized by a rotation of an angle θ. One finds [8] $\int d\theta\,\rho_\mu(G) = f(x) = \exp(-x)I_0(x)$, where $x = \mu\epsilon^2/8\sigma^2$, ϵ denoting the separation of the two coupled levels. For $x < 1$, $f(x) \approx 1$ and decreases as $1/\sqrt{x}$ for $x \gg 1$. This gives

$$\frac{E_\mu}{\Delta_2} = \frac{\sqrt{8\sigma^2/\mu}}{\Delta_2} \propto \frac{N}{\sqrt{\mu}}. \quad (4)$$

For $\epsilon < E_\mu$, one has the GOE statistics, while for $\epsilon \gg E_\mu$, the levels are uncorrelated.

(ii) $\Gamma > \Delta_2$.—Many neighboring G_{ii} are coupled by the off-diagonal terms. First, we consider the case where $\epsilon = |E_\alpha - E_\beta| < \Gamma$, i.e., the case where the two corresponding eigenvectors have a strong overlap. Assuming that the eigenvectors $|O_\alpha\rangle$ have nonzero coordinates of order $O_{n\alpha}^2 \approx \Delta_2/\Gamma$ over Γ/Δ_2 neighboring basis states only, one gets $\sum_{p=1}^N O_{p\alpha}^2 O_{p\beta}^2 \approx \Delta_2/\Gamma$, and $\exp(-\mu\epsilon^2\Delta_2/2\sigma^2\Gamma) \approx 1$, independent of ϵ ($<\Gamma$). Writing $O = \exp A$, with A a real antisymmetric matrix [$\mu(dO) = \prod_{\alpha<\beta} dA_{\alpha\beta}$], one can see that the small fluctuations of the $A_{n\alpha}$ around their typical values will not yield a correction to the GOE level repulsion. This means that there is no coupling between eigenvalues and eigenvectors as far as $\epsilon < \Gamma \equiv E_\mu$ with

$$\frac{E_\mu}{\Delta_2} \propto \frac{N^2}{\mu} \quad (5)$$

now, instead of $N/\sqrt{\mu}$ previously. When $|\epsilon| \gg \Gamma$, the eigenvectors do not overlap and the levels become uncorrelated. In Ref. [8], it was noted that if $O_{p\alpha} \approx \delta_{p,\alpha} + A_{p\alpha}$ where $A_{p\alpha} \ll 1$, $\sum_{p=1}^N O_{p\alpha}^2 O_{p\beta}^2 \approx 2A_{\alpha\beta}^2$, which gives a $1/|\epsilon|$ factor, after integration over $A_{\alpha\beta}$. This level attraction exactly compensates the level repulsion due to $\mu(dG)$. Qualitatively, one can adapt this reasoning to obtain the requested level attraction, after integration over the eigenvectors.

A numerical study of the GMPB ensemble as a function of μ exhibits the two regimes. The variance $\Sigma_2(E)$ of the number of levels in an interval E is shown in Fig. 2. For small E, $\Sigma_2(E)$ coincides with the GOE-logarithmic increase. One defines E_μ from the energy interval where Σ_2 is above the GOE curve by a certain threshold (e.g., 20%). For $E > E_\mu$, it turns out that $\Sigma_2(E)$ can be fitted by $(E/E_\mu)^{\alpha(\mu)}$, which gives a second determination of E_μ. Note that those methods give a nonzero E_μ (depending on the chosen threshold) even for uncorrelated levels, which has been subtracted from the data. The inset of Fig. 2 shows the agreement of the two determinations of E_μ and exhibits the predicted crossover for E_μ when $\Gamma \approx \Delta_2$ ($\mu \approx N^2$), from a N^2/μ dependence (small μ) towards a $N/\sqrt{\mu}$ dependence. The μ dependence of the exponent α (see inset in Fig. 4 below) depends on the exact form of the level interaction.

We now study directly the TIP Hamiltonian. For $d = 2$, Σ_2 is given around the band center (Fig. 3). We have obtained the same curves for $U = 1$ and $U = -1$, and consider in more detail repulsive interactions in rings containing 10×10 sites threaded by a magnetic flux, so that the level statistics should have a GUE behavior for energy intervals $E < E_U \equiv E_\mu$, with $(1 + \mu)^{-1} \approx 6U^2/L_1^{3d}B_1^2$. Except for this change from orthogonal to unitary symmetry, the similarity with the GMPB model is very striking. The crossover value $U_c \approx \sqrt{2}\,(8Vd + 4W)/\sqrt{\pi}\,L^{d/2}$ between the two regimes is of order 1, for the considered parameters. When $U > U_c$, we observe the U^2 behavior of E_U. However, the behaviors for large energy intervals and for large values of U differ. In the TIP Hamiltonian, the one particle level rigidity becomes relevant when $E > \Delta_1$ (one has Δ_1/Δ_2 superimposed GUE series when $U = 0$, correlations which are neglected in the GMPB model). The other difference results from the shifts UQ_{nn} of the diagonal terms which become important when U is large. Clearly, the GMPB model is oversimplified for giving the asymptotic regimes, for $E > \Delta_1$ and for large U, but gives the correct energy scale for $E_U < \Delta_1$, and its U dependence for not too large U. The U dependence of the exponent α (see inset of Fig. 4) is also different.

For $d = 1$, a similar study is very instructive. When $W = V = 1, E \sim 0$, we have $L_1 \approx 25$, which gives again $U_c \approx 1$. As expected, one can see in Fig. 4 that $E_U \propto |U|$ when $|U| < U_c$, but when $U > U_c$, the splitting of the energy band occurs, and E_U decreases. One recovers uncorrelated levels for very large U (for $d = 2$, there is only a saturation of E_U). For $d = 1$, this means that one can couple only two basis states within $L_1 \approx 25$, with a small enough value of U to justify the simplified GMPB ensemble. The observation of the U^2 behavior of E_U requires larger values of L_1 in $d = 1$ than considered in the numerical studies [4–6].

We now follow the argument developed by Imry [3] to estimate the localization length L_2. First, we consider a series of building blocks of size L_1^d. Γ_U is the smearing of the TIP levels of one of the blocks, due to the interaction-induced coupling with the neighboring block. For such a quasi-1D wire, the dimensionless conductance at scale L_1 is given by

$$g_2(L_1) \equiv \frac{L_2}{L_1} \approx \frac{1}{2} + A\frac{\Gamma_U}{\Delta_2}. \qquad (6)$$

The factor $1/2$ gives [5] the right limit when $U \to 0$ and A is a constant. Obviously, one should have $\Gamma_U \equiv E_U$. When $\Gamma_U > \Delta_2$, Γ_U is given by Fermi's golden rule, the case considered in Ref. [3], and we only discuss the

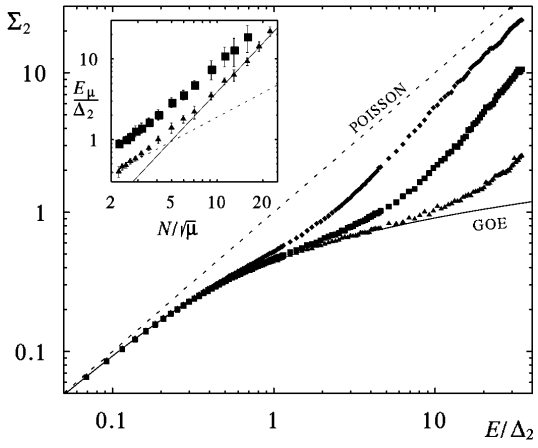

FIG. 2. Σ_2 for the GMPB ensemble ($N = 500$). Diamonds, squares, and triangles are for $\mu = 30\,000$, 5000, and 1000, respectively. The inset shows how E_μ depends on μ. The triangles give the energy where Σ_2 is 20% above the GOE value. The solid and the dotted line represent $E_\mu/\Delta_2 = 0.039N^2/\mu$ and $E_\mu/\Delta_2 = 0.19N/\sqrt{\mu}$, respectively. The squares result from a fit $\Sigma_2 = (E/E_\mu)^\alpha$, valid for $E \gg E_\mu$.

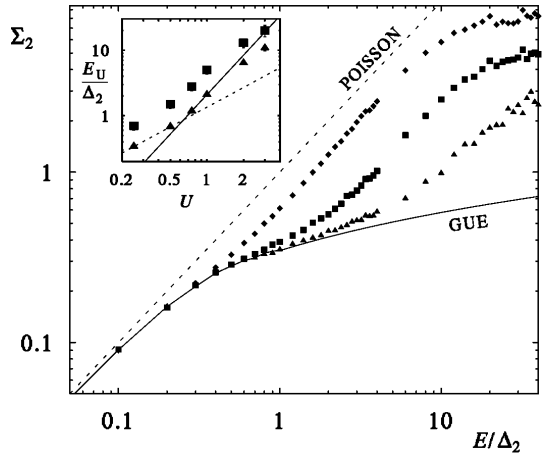

FIG. 3. Σ_2 for the symmetric states of a 2D TIP Hamiltonian (ring with 10×10 sites, $W = V = 1, E \sim 0$). The ring is threaded by a magnetic flux $\Phi = \Phi_0/4$. Diamonds, squares, and triangles are for $U = 0.25$, 0.75, and 2.0, respectively. The inset shows how E_U depends on U. The data are obtained as described in the caption of Fig. 2. The dotted line [solid line] corresponds to $E_U/\Delta_2 = 1.35U/V$ [$E_U/\Delta_2 = 2.1(U/V)^2$].

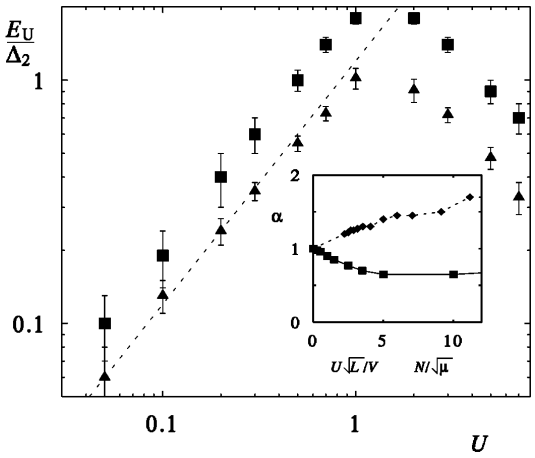

FIG. 4. E_U for a 1D TIP Hamiltonian ($W = V = 1$ and $L = L_1 = 25$ sites). The dotted line corresponds to $E_U/\Delta_2 = 1.2U/V$. Inset: α in the fit $\Sigma_2 \propto E^\alpha$ for large E. Squares: 1D TIP Hamiltonian, as a function of $U\sqrt{L}/V$. Diamonds: GMPB ensemble, as a function of $N/\sqrt{\mu}$.

case $\Gamma_U < \Delta_2$, where $\Gamma_U \approx \sqrt{U^2/L_1^{3d}}$. Physically, this means that U is so small that it couples only a single TIP state in one of the blocks to another TIP state in the next block, giving rise to "Rabi oscillations" between those two coupled states. The inverse lifetime is no longer given by the square of the coupling term, as in Fermi's golden rule, but by its absolute value. In addition, we have shown that this inverse lifetime also gives the scale below which one has a GOE spectral rigidity.

We discuss a few implications for TIP localization. For $d = 1$ and $U < U_c$ ($U_c \approx 1$ when $W \approx V \approx 1$, see Fig. 4), one gets $L_2/L_1 \approx 1/2 + A(|U|/B_1)\sqrt{L_1}$ which is in agreement with the dependence on U observed in the numerical studies [5]. The conjecture proposed in Ref. [5] gives L_1 instead of $\sqrt{L_1}$. As noted in Ref. [4], the distribution of the $Q_{ABA'B'}$ is far from being Gaussian. Moreover, the estimate of the variance comes from oversimplified "ergodic" one particle states [2]. This can matter as far as the description of the L_1 dependence by the GMPB model is concerned. However, the U dependence is not affected by these simplifications and is correctly described by the GMPB model.

If one considers two quasiparticles above a Fermi sea, one should replace [3] in Eq. (6) Δ_2 by $\Delta_2(E) \approx \Delta_1^2/E$, where E is the total excitation energy. One immediately obtains that the quasiparticle conductance $g_2^q(E, L_1)$ is of order of $g_2(L_1)$ when $E \approx B_1$, which gives $L_2^q(E \approx B_1) = L_2$, in agreement with Ref. [13]. Similarly, in three dimensions, Imry's relation $[E_{m2} \approx (B_1^2/|U|)E_{m1}^{\nu d/2}]$ between the one quasiparticle mobility edge E_{m1} and the two quasiparticle mobility edge E_{m2} does not change when $U < U_c$ (ν denotes the critical exponent associated with L_1).

In summary, we have shown that the basic concepts developed for noninteracting particles can be naturally extended to two interacting particles, after the changes $E_C \to E_\mu$ and $\Delta_1 \to \Delta_2$. A similar conclusion has been obtained from a nonlinear σ model description of the TIP Hamiltonian [14], when $L > L_1$. Moreover, our approach can be easily extended to an arbitrary number of particles.

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