Double-Cross Instability: An Absolute Instability Caused by Counter-Propagating Positive- and Negative-Energy Waves

A. J. Brizard, J. J. Morehead, and A. N. Kaufman*
Lawrence Berkeley National Laboratory, Berkeley, California 94720

E. R. Tracy
Department of Physics, College of William and Mary, Williamsburg, Virginia 23185
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The resonant interaction of a negative-energy wave with a positive-energy wave gives rise to a linear instability. Whereas a single crossing of rays in a nonuniform medium leads to a *convectively* saturated instability, we show that a double crossing can yield an *absolute* instability, if the two rays are oppositely directed. We obtain expressions for the growth rate and the threshold, and present one application. [S0031-9007(96)00921-0]

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Negative-energy waves are known to produce a number of interesting effects, such as linear instability due to dissipation, and explosive instability due to three-wave interaction [1]. Such waves are typically supported by beams [2] or flows [3], or by inverted populations [4]. Here we investigate a novel and possibly important phenomenon, the linear absolute instability caused by the double crossing, in ray phase space, of oppositely directed rays of a *negative-energy* wave and a *positive-energy* wave of equal frequencies. (An analogous phenomenon involving the transformation of a parametric convective instability into an absolute one because of wave reflection is studied by Cairns [5].)

In Fig. 1, we illustrate the situation to be studied. We assume that the medium has spatial variation in only *one* dimension x, so that each wave (of type $j = a$ or b), of definite frequency ω and wave-vector components (k_y, k_z), has the ray orbit $k_x^j(x)$, determined by its dispersion function $D_j(x, k_x; \omega, k_y, k_z) = 0$. Let a label the positive-energy wave, and b the negative-energy wave, of the same $(\omega; k_y, k_z)$. Near their caustics (where dk_x/dx is infinite), the two rays typically have different curvatures, and so can cross twice, as shown.

The motivation for our study is a plasma instability, wherein a magnetosonic wave is made unstable by an inverted population of neonatal alpha particles produced in fusion reactions [6,7]. This instability leads to enhanced emission at harmonics of the alpha gyrofrequency, and is considered a useful diagnostic [8]. Since the alphas can support a *negative-energy* Bernstein wave at a gyroharmonic [4], and since this wave can undergo *linear conversion* with a (positive-energy) magnetosonic wave when their frequencies and wave vectors are equal, the situation of Fig. 1 can arise. (Later in this Letter, we demonstrate that the rays are oppositely directed, and calculate the ray curvatures and other characteristics of this application.)

We first examine the energy transmission and conversion coefficients, T and C, at each of the crossings, and see how an absolute instability arises. At the lower crossing, energy conservation is represented as $T + C = 1$, where T is the ratio of the transmitted to incident flux of a, while C is the ratio of the b-converted to a-incident flux. When the wave energies have opposite signs, we have [9,10] $T = \exp(2\pi|\eta|^2/|\mathcal{B}|) > 1$, where η is the *coupling strength,* and $\mathcal{B}$ is the *Poisson bracket* of the two dispersion functions (evaluated at the conversion point): $\mathcal{B} \equiv \{D_a, D_b\}$ [see Eq. (8)]. Thus $C \equiv 1 - T < 0$, and if $T > 2$, then $|C| > 1$, i.e., ray b has *greater* energy flux magnitude than (incident) ray a.

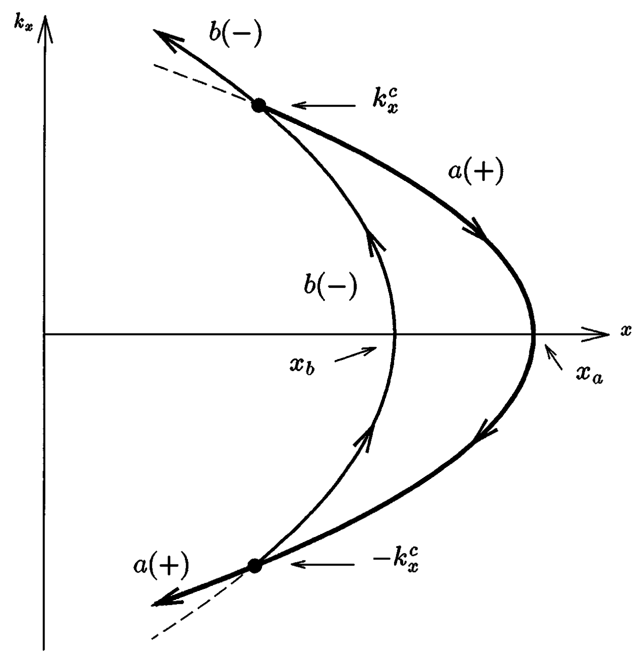

FIG. 1. The orbits of the *positive-energy* ray a (with *downward* velocity $\dot{k}_x^a < 0$) and the *negative-energy* ray b (with *upward* velocity $\dot{k}_x^b > 0$), of the same frequency ω. (The *signs* of $\dot{k}_x^j$ are chosen from the application discussed here.) Their caustics ($dk_x/dx \to \infty$) are at x_a, x_b. The rays *cross* and *convert* at $k_x = \pm k_x^c$. If the energy-flux magnitude in the converted ray *exceeds* that of the incident ray, at each crossing, the system is (absolutely) unstable.

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If there is only *one* crossing, that is the end of the story; but if the *second* (upper) crossing occurs (with the same parameters, for simplicity), then (converted) ray a arises with energy flux $|C|$ relative to the incident b, and thus $|C|^2$ relative to its value on entering the lower crossing. Thus the condition for absolute instability is $|C| > 1$, or $T > 2$, or

$$|\eta|^2 > [(\ln 2)/2\pi]\,|\mathcal{B}| \equiv |\eta_{\rm th}|^2. \tag{1}$$

We see that the *threshold* coupling strength (squared) is proportional to the Poisson bracket.

To evaluate $\mathcal{B}$, we examine the dispersion functions in more detail. In the neighborhood of a caustic, each ray ($j = a, b$) is a parabolic curve: $x(k_x) = x_j + \beta_j k_x^2$, where $x_j \equiv x(k_x = 0)$ is the caustic location, and $\beta_j \equiv dx/dk_x^2$ is the ray curvature. (As discussed later, we assume that D_j is *even* in k_x.) The two rays thus cross at $k_x = \pm(\Delta x/\Delta\beta)^{1/2} \equiv \pm k_x^c$, where $\Delta x \equiv x_a - x_b$ is the caustic separation, and $\Delta\beta \equiv \beta_b - \beta_a$ is the curvature difference. It follows that the dispersion functions have the form

$$D_j(x, k_x) = (D_x^j)\,(x - x_j - \beta_j k_x^2)\,, \tag{2}$$

where $D_x^j \equiv \partial D_j/\partial x = -(\partial D_j/\partial\omega)\,(\partial\omega/\partial x)_j = (\partial D_j/\partial\omega)\dot{k}_x^j < 0$ is proportional to the (constant) ray velocity in the k_x direction. We obtain (after straightforward algebra)

$$|\mathcal{B}| = 2D_x^a D_x^b (\Delta\beta) k_x^c \equiv 2D_x^a D_x^b (\Delta\beta\Delta x)^{1/2}. \tag{3}$$

Thus, for given parameters, the threshold coupling is minimized by *minimizing* the caustic separation Δx.

To determine the latter, we impose the requirement of phase matching: the phase change $\Delta\phi$ of ray a (say), after one circuit, must be an integer multiple N of 2π: $\Delta\phi = 2\pi N$. To find $\Delta\phi$, we use eikonal theory (to lowest order in η): $\Delta\phi = \oint x(k_x)\,dk_x - \pi/2$, where the first term is the standard phase integral (the area enclosed by the rays), and the second term is the sum of the two (lowest-order) phase shifts at the conversion points [11]. The evaluation of the phase integral is elementary: $\oint x(k_x)\,dk_x = (4/3)\,(\Delta\beta)^{-1/2}(\Delta x)^{3/2}$. The minimum separation $\Delta x_{\rm min}$ is thus obtained by setting $N = 0$:

$$\Delta x_{\rm min} = (3\pi/8)^{2/3}(\Delta\beta)^{1/3}. \tag{4}$$

Inserting (4) into (3), we have $|\mathcal{B}|_{\rm min} = (3\pi)^{1/3} D_x^a D_x^b \times (\Delta\beta)^{2/3}$, yielding the threshold from (1):

$$|\eta_{\rm th}|^2 = [(\ln 2)/2\pi]\,(3\pi)^{1/3} D_x^a D_x^b (\Delta\beta)^{2/3}. \tag{5}$$

To obtain the positive growth rate γ (when $T > 2$), we track the energy around one circuit, obtaining $\gamma = [\ln(T - 1)]/\tau$, where τ is the time interval for a circuit:

$$\tau = \sum_{j=a,b} \int dk_x/|\dot{k}_x^j| = (|\dot{k}_x^a|^{-1} + |\dot{k}_x^b|^{-1})\,(3\pi)^{1/3}(\Delta\beta)^{1/3}, \tag{6}$$

and (4) has been used.

We now proceed to derive and solve the differential equations for the propagation and coupling of the two wave fields. This will serve to check the validity of the results above based on eikonal theory. We begin with the variational Hermitian form $\mathcal{A}[\mathbf{E}] \equiv \int d^4x\,\mathbf{E}\cdot \mathrm{D}\cdot\mathbf{E}/2$, in terms of the total wave field $\mathbf{E}(\mathbf{x}, t)$ and the (given) dispersion operator $\mathrm{D}(x; \mathbf{k} = -i\nabla, \omega = i\partial_t)$. With D independent of (y, z, t), we take $\mathbf{E}(\mathbf{x}, t) = \mathbf{E}(x)\exp i(k_y y + k_z z - \omega t) + \text{c.c.}$, and obtain $\mathcal{A}[\mathbf{E}] = \int dx\,\mathbf{E}^*(x)\cdot\mathrm{D}\cdot\mathbf{E}(x)$, with $\mathrm{D}(x, k_x = -i\partial_x; k_y, k_z, \omega)$. Next we apply a congruent reduction [12]: From the three ($j = a, b, c$) eigenvalues $D_j(x, k_x)$ of the 3×3 matrix $\mathrm{D}(x, k_x)$, we select the two ($j = a, b$) whose dispersion curves $D_j(x, k_x) = 0$ have a double (avoided) crossing. *Outside* the crossings, we determine their polarizations $\hat{e}_j(x)$ [eigenvectors of the matrix $\mathrm{D}(x, k_x(x))$], and express $\mathbf{E}(x) = E_a(x)\hat{e}_a(x) + E_b(x)\hat{e}_b(x)$, where $\hat{e}_a, \hat{e}_b$ are now interpolated smoothly from the outside regions. Substituting, we now have

$$\mathcal{A}[E_a, E_b] = \int dx\left[\sum_{j=a,b} E_j^* D_j E_j + (E_a^*\eta E_b + \text{c.c.})\right], \tag{7}$$

where $D_j \equiv \hat{e}_j^*\cdot\mathrm{D}\cdot\hat{e}_j$, $\eta \equiv \hat{e}_a^*\cdot\mathrm{D}\cdot\hat{e}_b$. On varying $\mathcal{A}$, we obtain the coupled equations [9,10] for $E_a(x), E_b(x)$:

$$\begin{pmatrix} D_a & \eta \\ \eta^* & D_b \end{pmatrix}\begin{pmatrix} E_a \\ E_b \end{pmatrix} = 0\,. \tag{8}$$

Next, we find the *reference* frequency ω_0 such that $D_a(x, k_x; \omega_0) = 0 = D_b(x, k_x; \omega_0)$ are dispersion curves $x_j(k_x; \omega_0)$ whose caustics $x_j(\omega_0) \equiv x_j(k_x = 0; \omega_0) \equiv x_0$ *coincide* (i.e., $\Delta x = 0$). We expand $D_j(x, k_x; \omega)$ (to lowest order) about $x = x_0, k_x = 0, \omega = \omega_0$:

$$\begin{aligned} D_j(x, k_x; \omega) &= (\omega - \omega_0)D_\omega^j + (x - x_0)D_x^j + k_x^2 D_{k_x^2}^j \\ &= [\omega - \omega_0 + (x - x_0 - \beta_j k_x^2)\dot{k}_x^j]D_\omega^j\,. \end{aligned}$$

Expressing the *eigenfrequency* ω as $\omega - \omega_0 \equiv \Delta\omega + i\gamma$, this reads

$$\mathrm{D}_j(x, k_x; \omega) = [i\gamma + (x - x_j - \beta_j k_x^2)\dot{k}_x^j], \tag{9}$$

where $x_j \equiv x_j(\omega_0 + \Delta\omega) = x_0 + (\Delta\omega)\,(\partial x_j/\partial\omega)$ is the caustic at the real part of the eigenfrequency. Comparing (9) with (2), we note that $\gamma \equiv \mathrm{Im}\,\omega$ now appears explicitly. Because the field equations (8) are singular in the x representation, we Fourier transform to the k_x representation, with $x \to i\partial/\partial k_x$ in (9). We set the slowly varying η equal to its value at $(x_0, k_x = 0, \omega_0)$.

From the coupled equations, we now obtain the wave-energy *conservation* law

$$2\gamma \sum_j W_j(k_x) + \frac{\partial}{\partial k_x}\left(\sum_j \dot{k}_x^j W_j(k_x)\right) = 0\,, \qquad (10)$$

where $W_j(k_x) \equiv \omega D_\omega^j |E_j(k_x)|^2$ is the wave-energy density (in k_x space) of wave j. The first term is the time derivative of the total energy density $W(k_x) = W_a + W_b = W_a - |W_b|$, and the second is the divergence of the energy flux. This conservation law guides our heuristic interpretation of the double-conversion process.

The eigenfrequencies ω are determined by solving (8) numerically, subject to the boundary conditions representing zero *incoming* wave energy. The asymptotic behavior of the field magnitudes are, from (9) or (10), $|E_j(k_x)| \to A_\pm^j \exp(-\gamma k_x/\dot{k}_x^j)$, as $k_x \to \pm\infty$. Hence we require $A_+^a = 0 = A_-^b$ as boundary conditions (see Fig. 1). We now introduce the action-flux *amplitudes* $\overline{E}_j(k_x)$:

$$E_j(k_x) \equiv |D_x^j|^{-1/2}\,\overline{E}_j(k_x) \times \exp(-ix_j k_x - i\beta_j k_x^3/3 - \gamma k_x/\dot{k}_x^j)\,, \qquad (11)$$

the dimensionless variable $K \equiv (\Delta\beta)^{1/3} k_x$, the dimensionless eigenvalue

$$Z \equiv (\Delta\beta)^{-1/3}[\Delta x + i(|\dot{k}_x^a|^{-1} + |\dot{k}_x^b|^{-1})\gamma]\,, \qquad (12)$$

and the dimensionless coupling $\overline{\eta} \equiv (\Delta\beta)^{-1/3} \times (D_x^a D_x^b)^{-1/2}\eta$. The coupled field equations are then

$$\begin{pmatrix} -id/dK & \overline{\eta} e^{\Phi} \\ \overline{\eta}^* e^{-\Phi} & -id/dK \end{pmatrix} \begin{pmatrix} \overline{E}_a(K) \\ \overline{E}_b(K) \end{pmatrix} = 0\,, \qquad (13)$$

with $\Phi(K;Z) \equiv iZK - iK^3/3$. From (13), we see that the set of dimensionless complex eigenvalues $\{Z_N\}$ depends only on the *single* dimensionless parameter $|\overline{\eta}|^2$. (For given $\overline{\eta}$, we expect a discrete set of eigenvalues, ordered by their imaginary parts. This corresponds to the set of integers N.) Only that eigenvalue $Z_0(\overline{\eta})$ whose imaginary part is the largest is of interest. The threshold value ($|\overline{\eta}_{\rm th}|^2$) is thus found by setting $\mathrm{Im}\, Z_0(\overline{\eta}) = 0$. By numerical integration of (13), we obtain $|\overline{\eta}_{\rm th}|^2 = 0.236$. This is to be compared with the eikonal result (5): $|\overline{\eta}_{\rm th}|^2 = [(\ln 2)/2\pi](3\pi)^{1/3} D_x^a D_x^b (\Delta\beta)^{2/3}$, or $|\overline{\eta}_{\rm th}|^2 = 0.233$, in quite remarkable agreement.

By (12), the *real* part of the eigenvalue Z_0 (at threshold) is the dimensionless caustic separation. By (4), its analytic value is $\mathrm{Re}\, Z_0 = (3\pi/8)^{2/3} = 1.1$, to be compared with the numerical result 0.9, again in good agreement. Finally, we examine the dependence of the growth rate on the coupling, $d\gamma/d|\eta|^2$, or in dimensionless form, $d(\mathrm{Im}\, Z)/d|\overline{\eta}|^2$. Numerically, we find 2.9, to be compared with our analytic approximation, $(64\pi/9)^{1/3} = 2.8$, in excellent agreement. We conclude that the eikonal approximation is a reliable analytic solution.

As an application, we consider the double crossing of a *magnetosonic* wave a with a negative-energy α-particle *Bernstein* wave b, in a slab model of the outer edge of a tokamak: $\mathbf{B}_0 = \hat{z} B_0(x)$. We take $\omega \approx l\Omega_\alpha$, with $l^2 \gg 1$ (to model interpretations of ICE [8]; for simplicity, we choose deuterium as the majority species since $\Omega_D = \Omega_\alpha$), $k_z = 0$ (to avoid resonant-particle effects), and $f_\alpha \sim \delta(v_\perp - v_\alpha)$ (to avoid dispersion of cross-field drift). The polarizations are $\hat{e}_a = (-il\hat{x} + \hat{y})/(l^2+1)^{1/2}$, $\hat{e}_b = \hat{k}$, and the elements of the reduced dispersion matrix are

$$D_a(x, k_\perp^2; \omega) = \frac{c^2}{l^2}\left(\frac{1}{c_A^2(x)} - \frac{k_\perp^2}{\omega^2}\right), \qquad (14)$$

$$D_b(x, k_\perp^2; k_y, \omega) = -\frac{c^2}{l^2 c_A^2} - \frac{\omega_\alpha^2}{\omega\,\omega_l(x; k_y, \omega)}\,\mu_l(\lambda)\,, \qquad (15)$$

$$\eta(x, k_\perp^2; k_y, \omega) = i\,\frac{c^2}{l^2 c_A^2}\,\frac{\nu_l(\lambda)}{\mu_l(\lambda)}\,, \qquad (16)$$

$$\mu_l(\lambda) \equiv l^2\lambda^{-1}(J_l^2)', \qquad \nu_l(\lambda) \equiv \lambda^{-1}(l^2 J_l^2 - \lambda J_l J_l')', \qquad (17)$$

where $\lambda \equiv k_\perp v_\alpha/\Omega_\alpha = l v_\alpha/c_A$ is the Bessel-function argument, c_A is Alfvén speed, ω_α is plasma frequency of the alphas, and $\omega_l \equiv \omega - l\Omega_\alpha(x) - k_y v_D$, with cross-field drift v_D. In (14) and (15), we have used a cold-fluid model for the background plasma, which is justified because both $l^2 \gg 1$ and a large Doppler shift ($k_y v_D$) allow us to neglect deuterium kinetic gyroresonance effects.

From $D_j = 0$, the frequency functions are $\omega_a = k_\perp c_A$, $\omega_b = [l\Omega_\alpha(x) + k_y v_D][1 - \mu_l(\lambda)\omega_\alpha^2/\omega_M^2]$. The Bernstein wave b has negative energy if and only if $\mu_l(\lambda) < 0$ (which requires $\lambda > l$) [4]. For $k_x \ll |k_y|$, we express $k_\perp \equiv (k_y^2 + k_x^2)^{1/2} = |k_y| + k_x^2/2|k_y|$ (thus D_j is even in k_x, as assumed above). The ray velocities in k_x space are $\dot{k}_x^a = -\partial\omega_a/\partial x = -\omega/2L_n$ and $\dot{k}_x^b = -\partial\omega_b/\partial x = \omega/L_B$, where $L_n \equiv -(d\ln n_M/dx)^{-1} > 0$, $L_B \equiv -(d\ln B_0/dx)^{-1} > 0$, and $L_n \ll L_B$. Thus $\dot{k}_x^a/\dot{k}_x^b$ is negative as assumed, and $D_x^a D_x^b$ is positive. The ray curvatures are $\beta_a = -L_n/k_y^2$, $\beta_b \ll |\beta_a|$, so $\Delta\beta \approx |\beta_a|$. Inserting these formulas into (5), we obtain the threshold value of the (alpha/majority) density ratio:

$$\left(\frac{\omega_\alpha^2}{\omega_M^2}\right)_{\rm th} = (3\pi)^{1/3}[(\ln 2)/2\pi]\frac{|\mu_l|}{|\nu_l|^2} \times (|k_y|L_B)^{-1}(|k_y|L_n)^{1/3}, \qquad (18)$$

where $|k_y| = l\Omega_\alpha/c_A = l\omega_M/c$. The threshold is seen to be sensitive to the value of $|\mu_l(\lambda)|$, with λ dependent on the local majority density. Estimates based on JET parameters [8] indicate that this threshold may be exceeded (see [13] for additional details on this application), and thus an absolute instability of double-cross type may occur.

To summarize, we have shown that oppositely directed rays of positive- and negative-energy waves, in the neighborhood of their caustics, can produce an absolute instability from double-crossing linear conversion. We have obtained explicit expressions for the threshold and the growth rate.

This work can be extended in (at least) two directions. One is to allow spatial variation of the medium in more than one dimension. The other is (for plasma problems) to include kinetic effects (needed for $k_{\parallel} \neq 0$). This can be done by replacing the single collective negative-energy wave by a continuum of gyroballistic waves, a subset of which may have negative energy [4]. When this is done, a comparison can be made with the results obtained by more traditional methods [14,15].

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*Also at Department of Physics, University of California, Berkeley, CA 94720

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