Zero Temperature Chiral Phase Transition in SU(N) Gauge Theories

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We investigate the zero temperature chiral phase transition in an SU(N) gauge theory as the number of fermions N_f is varied. We argue that there exists a critical number of fermions N_f^c, above which there is no chiral symmetry breaking or confinement, and below which both chiral symmetry breaking and confinement set in. We estimate N_f^c and discuss the nature of the phase transition. [S0031-9007(96)00646-1]

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An SU(N) gauge theory, even at zero temperature, can exist in different phases depending on the number of massless fermions N_f in the theory. The phases are defined by whether or not chiral symmetry breaking takes place. For QCD with two or three light quarks, chiral symmetry breaking and confinement occur at roughly the same scale. By contrast, in any SU(N) gauge theory, asymptotic freedom (and hence chiral symmetry breaking and confinement) is lost if the number of fermions is larger than a certain value ($= 11N/2$ for fermions in the fundamental representation).

If the number of fermions N_f is reduced to just below $11N/2$, an infrared fixed point will appear, determined by the first two terms in the beta function. By taking the large N limit or by continuing to noninteger values of N_f [1], the value of the coupling at the fixed point can be made arbitrarily small, making a perturbative analysis reliable. Such a theory with a perturbative fixed point is a massless conformal theory. There is no chiral symmetry breaking and no confinement.

As N_f is reduced further, chiral symmetry breaking and confinement will set in. There have been lattice Monte Carlo studies of the N_f dependence of chiral symmetry breaking [2]. For example, Kogut and Sinclair [2] found that for $N = 3$ and $N_f = 12$ there is no chiral symmetry breaking, while Brown *et al.* [2] have found chiral symmetry breaking for $N = 3$ and $N_f = 8$. In this Letter we will estimate the critical value N_f^c at which this transition occurs. We then investigate the properties of the phase transition for $N_f \approx N_f^c$.

Our discussion will parallel an analysis of the chiral phase transition in QED3 and QCD3 [3,4]. In a large N_f expansion it was found that an appropriate effective coupling has an infrared fixed point with strength proportional to $1/N_f$, and that as N_f is lowered, the value of the fixed point exceeds the critical value necessary to produce spontaneous chiral symmetry breaking. It was argued that this critical value is large enough to make the $1/N_f$ expansion reliable.

An N_f dependence similar to the one we describe here has been found in $N = 1$ supersymmetric QCD [5]. This theory is not asymptotically free for large enough N_f, and has an infrared, conformal fixed point for a range of N_f below a certain value.

The Lagrangian of an SU(N) gauge theory is

$$\mathcal{L} = \overline{\psi}[i\not\partial + g(\mu)\not A^a T^a]\psi + \tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}, \qquad (1)$$

where ψ is a set of N_f four-component spinors, the T^a are the generators of SU(N), and $g(\mu)$ is the gauge coupling renormalized at some scale μ. The renormalization group (RG) equation for the running coupling is

$$\mu\frac{\partial}{\partial\mu}\alpha(\mu) = \beta(\alpha) \equiv -b\alpha^2(\mu) - c\alpha^3(\mu) - d\alpha^4(\mu) - \cdots, \qquad (2)$$

where $\alpha(\mu) = g^2(\mu)/4\pi$. With the N_f fermions in the fundamental representation, the first two coefficients are given by

$$b = \frac{1}{6\pi}(11N - 2N_f), \qquad (3)$$

$$c = \frac{1}{24\pi^2}\left(34N^2 - 10NN_f - 3\frac{N^2-1}{N}N_f\right). \qquad (4)$$

The theory is asymptotically free if $b > 0$ ($N_f < \frac{11}{2}N$). At two loops, the theory has an infrared stable, nontrivial fixed point if $b > 0$ and $c < 0$. In this case the fixed point is at

$$\alpha_* = -b/c\,. \qquad (5)$$

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Recall that the coefficients b and c are scheme independent [6], while the higher-order coefficients are scheme dependent. In fact one can always choose a renormalization scheme such that all the higher-order coefficients are zero; i.e., they can be removed by a redefinition of the coupling (change of renormalization scheme) $g' = g + G_1 g^3 + G_2 g^5 + \cdots$. Thus if a zero, α_*, of the β function exists at two loops, it exists to any order in perturbation theory [6]. Of course if the value of α_* is large enough, there could be important higher-order corrections to Green's functions of physical interest. Indeed, their perturbation expansion might not converge at all. In addition, nonperturbative effects, such as spontaneous chiral symmetry breaking, could eliminate even the existence of the fixed point. If the quarks develop a dynamical mass, for example, then below this scale only gluons will contribute to the β function, and the perturbative fixed point turns out to be only an approximate description, relevant above the chiral symmetry breaking scale.

For N_f sufficiently close to $11N/2$, the value of the coupling at the infrared fixed point can be made arbitrarily small. The RG equation for the running coupling can be written as

$$b \ln\left(\frac{q}{\mu}\right) = \frac{1}{\alpha} - \frac{1}{\alpha(\mu)} - \frac{1}{\alpha_*} \ln\left(\frac{\alpha[\alpha(\mu) - \alpha_*]}{\alpha(\mu)(\alpha - \alpha_*)}\right), \tag{6}$$

where $\alpha = \alpha(q)$. For $\alpha, \alpha(\mu) < \alpha_*$ we can introduce a scale defined by

$$\Lambda = \mu \exp\left[-\frac{1}{b\alpha_*} \ln\left(\frac{\alpha_* - \alpha(\mu)}{\alpha(\mu)}\right) - \frac{1}{b\alpha(\mu)}\right], \tag{7}$$

so that

$$\frac{1}{\alpha} = b \ln\left(\frac{q}{\Lambda}\right) + \frac{1}{\alpha_*} \ln\left(\frac{\alpha}{\alpha_* - \alpha}\right). \tag{8}$$

Then for $q \gg \Lambda$ the running coupling displays the usual perturbative behavior:

$$\alpha \approx \frac{1}{b \ln(q/\Lambda)}, \tag{9}$$

while for $q \ll \Lambda$ it approaches the fixed point α_*:

$$\alpha \approx \frac{\alpha_*}{1 + e^{-1}(q/\Lambda)^{b\alpha_*}}. \tag{10}$$

As N_f is decreased, the infrared fixed point α_* increases. We will suggest here that the breakdown of perturbation theory, described above, first happens due to the spontaneous breaking of chiral symmetry, and that the phase transition can be described by an RG improved ladder approximation of the Cornwall-Jackiw-Tomboulis (CJT) [7] effective potential. It is well known [8] that in vectorlike gauge theories the two-loop effective potential expressed as a functional of the quark self-energy becomes unstable to chiral symmetry breaking when the gauge coupling exceeds a critical value:

$$\alpha_c \equiv \frac{\pi}{3C_2(R)} = \frac{2\pi N}{3(N^2 - 1)}, \tag{11}$$

where $C_2(R)$ is the quadratic Casimir of the representation R. (A more general definition of the critical coupling is that the anomalous dimension of $\bar{\psi}\psi$ becomes 1 [9].) Thus we would expect that when N_f is decreased below the value N_f^c at which $\alpha_* = \alpha_c$, the theory undergoes a transition to a phase where chiral symmetry is spontaneously broken. The critical value N_f^c is given by

$$N_f^c = N\left(\frac{100N^2 - 66}{25N^2 - 15}\right). \tag{12}$$

For large N, N_f^c approaches $4N$, while for $N = 3$, N_f^c is just below 12. Note that this is consistent with lattice QCD results [2], which suggest that $8 < N_f^c \leq 12$.

Is this simple analysis reliable? After all, it could be that when α_* is as large as α_c the perturbative expansion for the CJT potential has broken down. To address this question we provide a crude estimate of the higher-order corrections to the CJT potential. An explicit computation of the next-to-leading term (or equivalently the next-to-leading term in the gap equation) [10] for $\alpha_* \approx \alpha_c$ produces an additional factor of approximately $\epsilon = \alpha_c N/4\pi$. This is the factor remaining after the appropriate renormalizations are absorbed into the definition of the coupling constant. From Eq. (11) we see that

$$\epsilon = \frac{1}{6(1 - 1/N^2)}. \tag{13}$$

For QCD, $\epsilon \approx 0.19$. If higher orders in the computation produce approximately this factor, the perturbative expansion of the CJT potential may be reliable. (It is worth noting that in condensed matter physics one can often (though not always) obtain useful information from the Wilson-Fisher expansion in a parameter that is set to 1 at the end of the calculation.) The same may be true of various Green's functions encountered in the skeleton expansion of the CJT potential.

We next explore the nature of the chiral phase transition at $N_f = N_f^c$ and its relation to confinement. It is useful to consider first the behavior in the broken phase $N_f < N_f^c$ ($\alpha_* > \alpha_c$). Here each quark develops a dynamical mass $\Sigma(p)$. For $N_f \to N_f^c$ from below ($\alpha_* \to \alpha_c$ from above), $\Sigma(p)$ can be determined by solving a linearized Schwinger-Dyson gap equation in ladder approximation. For momenta small compared to Λ, the effective coupling strength is α_*, while for momenta above Λ it falls according to Eq. (9). The resulting solution for $\Sigma(0)$ is [11]

$$\Sigma(0) \approx \Lambda \exp\left(-\frac{\pi}{\sqrt{\alpha_*/\alpha_c - 1}}\right). \tag{14}$$

The behavior of $\Sigma(p)$ as a function of p will be discussed shortly.

Once the dynamical mass $\Sigma(p)$ is formed, the fermions decouple below this scale, leaving the pure gauge theory behind. One might worry that this would invalidate the above gap equation analysis since it relies on the fixed point which exists only when the fermions contribute to the β function. This is not a problem, however, since it can be shown that when $\Sigma(0) \ll \Lambda$ the dominant momentum range in the gap equation, leading to the exponential behavior of Eq. (14), is $\Sigma(0) < p < \Lambda$. In this range, the fermions are effectively massless and the coupling does appear to be approaching an infrared fixed point. Note that the condition $\Sigma(0) \ll \Lambda$ is indeed satisfied for N_f sufficiently close to N_f^c.

Below the scale $\Sigma(0)$ the quarks can be integrated out; thus the effective β function has no fixed point and the gluons are confined. The confinement scale can be estimated by noting that at the quark decoupling scale $\Sigma(0)$, the effective coupling constant is of order α_c. A simple estimate using Eqs. (2)–(4) then reveals that the confinement scale is roughly the same order of magnitude as the chiral symmetry breaking scale. When N_f is reduced sufficiently below N_f^c so that α_* is not close to α_c, both $\Sigma(0)$ and the confinement scale become of order Λ. The linear approximation to the gap equation will then no longer be valid, and it will probably no longer be the case that higher-order contributions to the effective potential can be argued to be small.

It is interesting to compare the behavior of the broken phase for N_f near N_f^c to the walking technicolor gauge theories discussed recently in the literature [12]. We have argued here that for N_f just below N_f^c, the dynamical breaking is governed by a linearized ladder gap equation with a coupling α_* just above α_c. As the momentum p increases, $\alpha(p)$ stays near α_* (it "walks") until p becomes of order Λ, and only falls above this scale. It can then be seen [13] that the dynamical mass $\Sigma(p)$ falls as $1/p$ (i.e., the anomalous dimension of $\overline{\psi}\psi$ is ≈ 1) for $\Sigma(0) < p < \Lambda$ and only begins to fall more rapidly (as $1/p^2$) at larger momenta. This is precisely the walking behavior employed in technicolor theories and referred to there as high momentum enhancement. In that case, however, there was no IR fixed point to keep the β function near zero and slow the running of the coupling. It was noted instead that the same effect would emerge if the β function was small at each order by virtue of partial cancellations between fermions and bosons.

From the smooth behavior of the order parameter $\Sigma(0)$ [Eq. (14)], it would naively appear that the chiral phase transition at $N_f = N_f^c$ ($\alpha_* = \alpha_c$) is second order. In this Letter we will use the phrase "second order" to refer exclusively to a phase transition where the correlation length diverges as the critical point is approached from either side. In other words, there is a light excitation coupling to the order parameter that becomes massless at the critical point. In the broken phase, this mode would be present along with the massless Goldstone modes. In the symmetric phase, all these modes would form a light, degenerate multiplet, becoming massless at the critical point [14].

We examine the correlation length by working in the symmetric phase and searching for poles in the (flavor and color-singlet) quark-antiquark scattering amplitude, computed in the same (RG improved, ladder) approximation leading to Eq. (14). The analysis is similar to that carried out for QED3 [4]. If the transition is second order, then at least one pole should move to zero momentum as we approach the critical point (i.e., the correlation length should diverge). We take the incoming (Euclidean) momentum of the initial quark and antiquark to be $q/2$, but keep a nonzero momentum transfer by assigning outgoing momenta $q/2 \pm p$ for the final quark and antiquark. Any light scalar resonances should make their presence known by producing a pole in the scattering amplitude (when continued to Minkowski q^2).

If the Dirac indices of the initial quark and antiquark are λ and ρ, and those of the final state quark and antiquark are σ and τ, then the scattering amplitude can be written (for small q) as $T_{\lambda\rho\sigma\tau}(p,q) = \delta_{\lambda\rho}\delta_{\sigma\tau}T(p,q)/p^2 + \cdots$, where the $\cdots$ indicates pseudoscalar, vector, axial-vector, and tensor components, and we have factored out $1/p^2$ to make $T(p,q)$ dimensionless. We contract Dirac indices so that we obtain the Bethe-Salpeter equation for the scalar s-channel scattering amplitude $T(p,q)$, containing only t-channel gluon exchanges. If $p^2 \gg q^2$, then q^2 will simply act as an infrared cutoff in the loop integrations. The Bethe-Salpeter equation in the scalar channel for $p \ll \Lambda$ is

$$T(p,q) \approx \frac{\alpha_*}{\alpha_c}\pi^2 + \frac{\alpha_*}{4\alpha_c}\int_{q^2}^{p^2} dk^2\, T(k,q)\frac{1}{k^2} + \frac{\alpha_*}{4\alpha_c} \times \int_{p^2}^{\Lambda^2} dk^2\, T(k,q)\frac{p^2}{k^4}, \qquad (15)$$

where Λ is the scale introduced in Eq. (7). [Note that contributions from the integration region $k^2 > \Lambda^2$ are suppressed by a factor p^2/Λ^2, and a falling $\alpha(k)$.] The first term in Eq. (15) is simply one gluon exchange. We have used Landau gauge ($\xi = 1$) where the quark wave-function renormalization vanishes to lowest order. Because of the existence of the fixed point, it is a good approximation to have replaced $\alpha(p)$ and $\alpha(p-k)$ by α_* at momentum scales below Λ.

For momenta $p^2 > q^2$, Eq. (15) can be converted to a differential equation with appropriate boundary conditions. The solutions have the form

$$T(p,q) = A(q)\left(\frac{p^2}{\Lambda^2}\right)^{\frac{1}{2}+\frac{1}{2}\eta} + B(q)\left(\frac{p^2}{\Lambda^2}\right)^{\frac{1}{2}-\frac{1}{2}\eta}, \qquad (16)$$

where $\eta = \sqrt{1 - \alpha_*/\alpha_c}$. The coefficients A and B can be determined by substituting this solution back into

Eq. (15). This gives

$$A = \frac{-2\pi^2(1-\eta)^2}{(1+\eta)} \frac{(q^2/\Lambda^2)^{-\frac{1}{2}+\frac{1}{2}\eta}}{1-[(1-\eta)/(1+\eta)]^2(q^2/\Lambda^2)^\eta}, \tag{17}$$

$$B = \frac{2\pi^2(1-\eta)\,(q^2/\Lambda^2)^{-\frac{1}{2}+\frac{1}{2}\eta}}{1-[(1-\eta)/(1+\eta)]^2(q^2/\Lambda^2)^\eta}. \tag{18}$$

Note that there is an infrared divergence in the limit $q^2 \to 0$ in both Eqs. (17) and (18). That this is an infrared divergence rather than a pole corresponding to a bound state can be seen from the fact that the divergence exists for arbitrarily weak coupling ($\alpha_* \to 0$). In fact, it can already be seen at order α_*^2 in the one-loop (two gluon exchange) diagram. As required by the Kinoshita-Lee-Nanenberg theorem [15], this infrared divergence will be canceled in a physical scattering process by the emission of soft quanta.

If we denote the location of the poles of the functions A and B in the complex q^2 plane by q_0^2, we have

$$|q_0^2| = \Lambda^2\left(\frac{1+\eta}{1-\eta}\right)^{2/\eta}. \tag{19}$$

We see that there is no pole that approaches the origin $q_0^2 = 0$ as $\alpha_* \to \alpha_c$. Thus the correlation length does not diverge, and the transition is not second order. (It is not conventionally first order either since the order parameter vanishes continuously at the critical point.) Note that the behavior of the zero temperature chiral phase transition is different from the finite temperature case due to the presence of long-range gauge forces. At finite temperatures, gluons are screened, and thus there are only short-range forces present and only conventional first or second order transitions are possible.

To conclude, we have argued that as the number of quark flavors, N_f, is reduced, QCD-like theories in four dimensions undergo a chiral phase transition at a critical value N_f^c [Eq. (12)]. For $N_f < N_f^c$, chiral symmetries are spontaneously broken, while they are unbroken for $N_f > N_f^c$. We have explored the nature of the chiral phase transition, arguing that it can be described using the QCD gap equation in ladder approximation (equivalently the two-loop approximation to the CJT potential). We have also argued that even though the order parameter vanishes at the critical point, the correlation length does not diverge (i.e., the phase transition is not second order). The critical behavior described here is similar to that found in QED3 and QCD3 [3,4]. We have, of course, not proven that higher-order corrections to our computation are small. Further study of this question as well as lattice Monte Carlo studies of the zero temperature phase transition would help to confirm or disprove our conclusions.

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