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Stability Criterion for Dark Solitons

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We prove that the inequality $\partial P/\partial v < 0$ is the stability criterion for dark solitons of the one-dimensional nonlinear Schrödinger equation with general nonlinearity. Here v is the soliton's velocity, and P renormalized momentum: $P = (i/2)\int(\psi_x^*\psi - \psi_x\psi^*)(1 - \rho_0/|\psi|^2)\,dx$. [S0031-9007(96)00864-2]

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The *static* dark solitons of the nonlinear Schrödinger equation (NLS) can be classified under two broad classes. *Bubbles* are one-, two-, and three-dimensional *nontopological* solitons arising typically in models with competing interactions [1,2]. The static bubbles are always unstable [1–3], and this property endows them with a transparent physical interpretation as nuclei of the first order phase transitions. Recent experimental results where vapor bubbles in superfluid ^{4}He were nucleated by a decompression wave [4], stimulated application of the NLS to the dynamical cavitation [5]. The second class includes topological solitons of the Gross-Pitayevski equation (alias repulsive NLS on the plane), and their one-dimensional counterparts, kinks. The repulsive NLS equation provides a semiclassical description of neutral superflows and its vortex solutions correspond to quantum vortices. The current upsurge of interest in the NLS description of ^{4}He is due to its capability of capturing details of intricate dynamical mechanisms such as vortex nucleation and vortex-sound interaction [6].

The above classification (topological vs nontopological) is difficult to extend to *traveling* dark solitons which always have some topological properties (the phase approaches different values at different spatial infinities). One-dimensional traveling dark solitons have been experimentally observed in optical fibers [7,8]; it was noticed that dark solitons are less influenced by noise, and their interaction is weaker than that of bright solitons. These facts suggest the feasibility of the use of dark solitons in optical communications [8]. A lot of interest in optics is also attracted by the *spatial* dark solitons. Physically, spatial dark soliton is a self-trapped dark band (or a more complicated pattern in the case of higher-dimensional spatial soliton) superimposed on an otherwise uniform background illumination where the self-defocusing is balanced by the diffraction of the band. Mathematically, it is a solution of the very same NLS equation where t denotes the propagation coordinate. Spatial solitons were observed in Kerr and photovoltaic media [9]; the envisioned applications include optical encoding, limiting, switching and computing, and nonlinear filtering [9,10].

The stability properties of dark solitons play the key role in all these applications. It turned out that stability of the dark soliton is determined by its velocity. Namely, numerical simulations of traveling 1D bubbles [11] revealed the existence of a critical velocity $v_{\rm cr}$ such that the bubble is stable for $v > v_{\rm cr}$ and unstable otherwise. This property was confirmed by the numerical analysis of the spectrum of linearized excitations about the soliton [12]. Later it was observed [13] that the region $v > v_{\rm cr}$ is well described by the inequality $\partial P/\partial v < 0$, where

$$P = \frac{i}{2}\int_{-\infty}^{\infty}(\psi_x^*\psi - \psi_x\psi^*)\,dx - \rho_0 \mathrm{Arg}\,\psi\,\Big|_{-\infty}^{+\infty} = \frac{i}{2}\int_{-\infty}^{\infty}(\psi_x^*\psi - \psi_x\psi^*)\left(1 - \frac{\rho_0}{|\psi|^2}\right)dx \qquad (1)$$

is the renormalized momentum. However, apart from the observation that the criterion $\partial P/\partial v < 0$ yields a condition of the minimality of energy on trial functions of some special form [14], no proof of it has been given so far. Consequently, both the accuracy and range of validity of this criterion has remained an open question. In this Letter we propose such a proof.

We consider the NLS equation with a general nonlinearity

$$i\psi_t + \psi_{xx} + F(|\psi|^2)\psi = 0\,, \tag{2}$$

where $F(\rho_0) = 0$ for some positive constant ρ_0. The traveling dark soliton is a localized solution with $\psi(x,t) = \phi(\tilde{x})$ where $\tilde{x} = x - vt$ and $\phi(x)$ satisfies

$$-iv\phi_x + \phi_{xx} + F(|\phi|^2)\phi = 0\,, \tag{3}$$

together with the nonvanishing boundary conditions

$$\phi(x) \longrightarrow \sqrt{\rho_0}\, e^{\mp i\mu} \quad \text{as } x \longrightarrow \pm\infty\,. \tag{4}$$

Equation (2) with the boundary conditions (4) has three integrals of motion: "number of particles" (or "complementary power") $N = \int_{-\infty}^{\infty}(|\psi|^2 - \rho_0)\,dx$; energy

$$E = \int_{-\infty}^{\infty}\{|\psi_x|^2 + U(|\psi|^2)\}\,dx\,, \tag{5}$$

where $U(\rho) = -\int_{\rho_0}^{\rho} F(\rho)\,d\rho$; and momentum Eq. (1). The last term in Eq. (1) is essential for solutions with nonvanishing boundary conditions. The addition of this term makes P functionally differentiable and therefore compatible with the Hamiltonian structure of the model [14]. (For Lagrangian formulation, see [15].)

Stability.—We first show that when $\partial P/\partial v < 0$, the dark soliton is stable. To this end, we construct the Liapunov functional out of two constants of motion:

$$\mathcal{L}[\psi] = E - E_s - v(P - P_s) + \frac{\alpha}{4}(P - P_s)^2. \tag{6}$$

Here $E_s = E[\phi]$ and $P_s = P[\phi]$ are the energy and momentum of the dark soliton $\phi(\tilde{x})$. We did not include the number of particles since, as it was demonstrated in [1,2,14], N is irrelevant for the stability analysis.

For small deviations $\delta\psi(\tilde{x},t) = \psi(x,t) - \phi(\tilde{x})$, we have

$$\mathcal{L}[\psi] = (\delta\Psi, \Lambda\delta\Psi)\,, \tag{7}$$

where $\delta\Psi$ is a two-component real vector composed of the real and imaginary part of $\delta\psi$: $\delta\Psi = (\mathrm{Re}\,\delta\psi, \mathrm{Im}\,\delta\psi)^T$, and the integro-differential operator Λ is defined as

$$\Lambda\delta\Psi = H\delta\Psi + \alpha(J\Phi', \delta\Psi)J\Phi'. \tag{8}$$

Here $' \equiv d/dx$; the vector Φ is constructed out of ϕ_R and ϕ_I: $\Phi = (\phi_R, \phi_I)^T$, where $\phi_R(x) + i\phi_I(x) = \phi(x)$, and H is a matrix differential operator

$$H = -\frac{d^2}{dx^2}I + v\frac{d}{dx}J + V(x)\,, \tag{9}$$

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{10}$$

$$V(x) = -\begin{pmatrix} F + 2F_\rho\phi_R^2 & 2F_\rho\phi_R\phi_I \\ 2F_\rho\phi_R\phi_I & F + 2F_\rho\phi_I^2 \end{pmatrix}, \tag{11}$$

where $\rho = \phi_R^2 + \phi_I^2$, and I the identity matrix. Finally, α is a positive constant which will be fixed later on.

In order that the soliton be stable, Λ should be positive definite; thus we need to examine eigenvalues of Λ:

$$\Lambda\delta\Psi = \lambda\delta\Psi\,. \tag{12}$$

It is instructive, however, to start with eigenvalues of the operator H. It has two zero eigenvalues: $H\Phi' = HJ\Phi = 0$, and exactly one negative eigenvalue η:

$$HZ(x) = \eta Z(x), \qquad \eta < 0\,. \tag{13}$$

To prove the latter, we first notice that when $v = 0$, we can choose $\phi_I(x) \equiv 0$ and H becomes diagonal:

$$H = \begin{pmatrix} L_0 & 0 \\ 0 & L_1 \end{pmatrix}, \tag{14}$$

where the Schrödinger operators L_0 and L_1 have obvious zero modes: $L_0\phi' = (-\partial^2 - F - 2F_\rho\phi^2)\phi' = 0$; $L_1\phi = (-\partial^2 - F)\phi = 0$. As we mentioned, the quiescent dark solitons can be classified as either kinks or bubbles. Kink is a monotonically growing real function, with $\phi(0) = 0$. Since $\phi'(x)$ is nowhere vanishing, the zero eigenvalue of L_0 is the minimum eigenvalue. On the other hand, since $\phi(x)$ is a zero mode of L_1 and it has one node, L_1 has (one) eigenvalue $\eta < 0$ with a nodeless eigenfunction. The case of bubbles is similar; the quiescent bubble is a nowhere vanishing real solution, with just one extremum: $\phi'(0) = 0$. Hence L_1 is positive definite while L_0 has only one negative eigenvalue η. Thus in both cases H has one and only one negative eigenvalue.

As v deviates from zero, the negative eigenvalue η of the operator H will move along the real axis, but can never pass through the origin. Indeed, assume $\eta \longrightarrow 0$, with the corresponding eigenfunction $Z(x)$ approaching some $Z_0(x)$. For all v the eigenfunction $Z(x)$ will be orthogonal to both $\Phi'(x)$ and $J\Phi(x)$, since eigenfunctions of a Hermitean operator pertaining to different eigenvalues are orthogonal. By continuity $(Z_0, \Phi') = (Z_0, J\Phi) = 0$ follows and so Z_0 is linearly independent of Φ' and $J\Phi$. However, the null space of H is spanned by the eigenfunctions Φ' and $J\Phi$ and therefore there can be no third independent eigenfunction with zero eigenvalue [12]. Hence η cannot approach zero. For the same reason no additional negative eigenvalues can emerge when v is varied.

Thus the operator $H - \lambda I$ is invertible for all $\lambda \le 0$ such that $\lambda \ne \eta$, and Eq. (12) becomes, for these λ,

$$\delta\Psi = -\alpha(J\Phi', \delta\Psi)(H - \lambda I)^{-1}J\Phi'.$$

Here we have assumed that $(J\Phi', \delta\Psi) \ne 0$; the converse will be discussed below. Taking the scalar product with $J\Phi'$, we obtain

$$g(\lambda) \equiv \alpha\big(J\Phi', (H - \lambda I)^{-1}J\Phi'\big) + 1 = 0\,. \tag{15}$$

When $\lambda \to \eta \pm 0$, the function $g(\lambda) \to \mp\infty$. This can be observed by decomposing $J\Phi'$ into parts parallel and orthogonal to $Z(x)$: $J\Phi'(x) = CZ(x) + Y(x)$, where $C = \text{const}$ and $(Z, Y) = 0$. Then

$$(H - \lambda I)^{-1} J\Phi'(x) = \frac{CZ(x)}{\eta - \lambda} + \sigma(x),$$

where $\sigma = (H - \lambda I)^{-1} Y$. The vector $Y(x)$ belongs to the invariant subspace of H, which is orthogonal to the eigenfunction $Z(x)$. The vector $\sigma(x)$ is in the same subspace and therefore $(\sigma, Z) = 0$. The function $g(\lambda)$ becomes

$$g(\lambda) = \frac{\alpha C^2}{\eta - \lambda} + \alpha\big(Y, (H - \lambda I)^{-1} Y\big) + 1, \tag{16}$$

where the second term is finite and positive for all $\lambda \leq 0$. Sending $\lambda \to \eta \pm 0$, we get $g(\lambda) \to \mp\infty$.

Next, since $J\Phi'$ is orthogonal to both zero modes of H, i.e., to Φ' and $J\Phi$, the function $H^{-1}J\Phi'$ is bounded. More precisely, comparing to the relation

$$H \partial\Phi/\partial v = -J\Phi', \tag{17}$$

which arises by differentiating Eq. (3), we have $H^{-1}J\Phi' = -\partial\Phi/\partial v$. Hence,

$$g(0) = -\alpha\left(J\Phi', \frac{\partial\Phi}{\partial v}\right) + 1 = \frac{\alpha}{2}\frac{\partial P}{\partial v} + 1, \tag{18}$$

where we have used the fact that $2(\Phi', J\partial\Phi/\partial v) = \partial P/\partial v$. Assuming $\partial P/\partial v < 0$ and choosing α sufficiently large, we can always ensure that $g(0) < 0$. Consequently, since $g(\lambda)$ is a monotonically growing function of λ:

$$\frac{dg}{d\lambda} = \alpha\big(J\Phi', (H - \lambda I)^{-2} J\Phi'\big) > 0, \tag{19}$$

Eq. (15) cannot be satisfied for any $\lambda < 0$ and the operator Λ does not have negative eigenvalues.

It remains for us to consider perturbations $\delta\Psi$ with $(J\Phi', \delta\Psi) = 0$. In this case the minimum of the functional $\mathcal{L} = \mathcal{L}[\phi + \delta\psi]$ occurs at solutions to the equation

$$H\delta\Psi = \lambda\delta\Psi + \beta J\Phi', \tag{20}$$

where β is the Lagrange multiplier. Solving for $\delta\Psi$ and substituting into $(J\Phi', \delta\Psi) = 0$ yields $\big(J\Phi', (H - \lambda I)^{-1} J\Phi'\big) = 0$. From the above analysis it follows, however, that this equation cannot be satisfied for $\lambda < 0$, and so $\min \mathcal{L} = \lambda(\delta\Psi, \delta\Psi) > 0$.

Thus the functional $\mathcal{L}[\psi]$ is positive definite [16] and the soliton is stable for $\partial P/\partial v < 0$. This inequality is *a priori* satisfied when the velocity v approaches the velocity of sound waves, $v \to c = \sqrt{2\rho_0 U_{\rho\rho}(\rho_0)}$. In this transonic limit the moving dark soliton is described [1] by the soliton of the Korteweg–de Vries equation which is given by an explicit formula; it is straightforward to verify that it satisfies $\partial P/\partial v < 0$. When the velocity goes down from c, the derivative $\partial P/\partial v$ can become positive; this happens, for instance, for the bubbles [1,2,12,14]. Let us show that in the region $\partial P/\partial v > 0$, the dark soliton is unstable.

Instability.—We consider a neighborhood of the critical velocity $v = v_{\rm cr}$ where the derivative $\partial P/\partial v$ changes its sign. Linearizing Eq. (2) about the soliton $\phi(\tilde{x})$ and assuming that, for $v < v_{\rm cr}$, the perturbation $\delta\Psi(\tilde{x}, t)$ depends on time as $\delta\Psi = Z(\tilde{x})e^{\lambda t}$, we arrive at the eigenvalue problem

$$HZ(x) = \lambda JZ(x), \qquad Z(\pm\infty) = 0, \tag{21}$$

with H and J as in (9) and (10), respectively. Letting $v = v_{\rm cr} + \epsilon$, we can expand Φ in powers of ϵ: $\Phi(x; v) = \Phi_0(x) + \epsilon\Phi_1(x) + \epsilon^2\Phi_2(x) + \cdots$. Accordingly, the operator H expands as $H = H_0 + \epsilon H_1 + \epsilon^2 H_2 + \cdots$. *A priori*, it is not obvious how λ should scale as $v \to v_{\rm cr}$. In the case of the cubic-quintic nonlinearity, numerical analysis suggests that $\lambda \sim \epsilon$ [12]; in general, however, other powers of ϵ cannot be ruled out. Notice that Eq. (21) is an eigenvalue problem for a non-Hermitean operator, $J^{-1}H$, and so λ does not have to be analytic in ϵ. However, the only admissible nonanalytic scaling is $\lambda \sim \epsilon^{1/2}$; other fractal powers would not do. (This follows from the fact that eigenvalues of the operator $J^{-1}H$ must always come in quadruplets: $\lambda, -\lambda, \lambda^*$, and $-\lambda^*$ [17].) Consequently, in general λ expands as

$$\lambda = \epsilon^{1/2}\lambda_1 + \epsilon\lambda_2 + \epsilon^{3/2}\lambda_3 + \cdots, \tag{22}$$

and the corresponding eigenfunction $Z(x)$ as

$$Z(x) = Z_0(x) + \epsilon^{1/2}Z_1(x) + \epsilon Z_2(x) + \cdots. \tag{23}$$

Substituting the above expansions into (21), we obtain a chain of equations:

$$H_0 Z_0 = 0, \tag{24}$$

$$H_0 Z_1 = \lambda_1 J Z_0, \tag{25}$$

$$H_0 Z_2 = J(\lambda_2 Z_0 + \lambda_1 Z_1) - H_1 Z_0, \tag{26}$$

$$H_0 Z_3 = J(\lambda_3 Z_0 + \lambda_2 Z_1 + \lambda_1 Z_2) - H_1 Z_1, \tag{27}$$

$$H_0 Z_4 = J(\lambda_2 Z_2 + \lambda_1 Z_3 + \lambda_3 Z_1 + \lambda_4 Z_0) - H_1 Z_2 - H_2 Z_0. \tag{28}$$

In order to solve these equations, we will make use of a series of useful identities. Expanding Eq. (17) in ϵ gives

$$H_0\Phi_1 = -J\Phi_0', \tag{29}$$

$$2H_0\Phi_2 + H_1\Phi_1 = -J\Phi_1', \tag{30}$$

while the expansion of $H\Phi' = HJ\Phi = 0$ produces

$$H_1\Phi_0' + H_0\Phi_1' = 0, \tag{31}$$

$$H_0\Phi_2' + H_1\Phi_1' + H_2\Phi_0' = 0, \tag{32}$$

$$H_0 J\Phi_1 + H_1 J\Phi_0 = 0 . \quad (33)$$

The only asymptotically decaying solution to Eq. (24) is $Z_0(x) = \Phi_0'$. Substituting into (25) and comparing to (29), we get $Z_1(x) = -\lambda_1\Phi_1(x)$. (This solution is defined up to the addition of a linear combination of Φ_0' and $J\Phi_0$ but these terms cancel in all scalar products below.) It is important to notice that the first-order correction $Z_1(x)$ tends to a nonzero constant vector column as $x \to \pm\infty$. This means that for large $|x|$ we have $\epsilon Z_1^2(x) \gg Z_0^2(x)$ and the hierarchy of Eqs. (24)–(28) is not valid. In particular, Eq. (25) should be replaced, for $x \to \pm\infty$, by

$$H_0 Z_1 = \epsilon^{1/2}\lambda_1 J Z_1 , \quad (34)$$

the solution of which decays as

$$Z_1(x) \longrightarrow \begin{pmatrix} u^{\pm} \\ v^{\pm} \end{pmatrix} \exp(k_{\pm}x), \qquad x \longrightarrow \pm\infty , \quad (35)$$

where $k_{\pm} = \epsilon^{1/2}\lambda_1/(v \mp c) + O(\epsilon^{3/2})$, and $u_{\pm}/v_{\pm} = \pm\tan\mu + O(\epsilon^{1/2})$. Since $\mathrm{Re}\,\phi_v/\mathrm{Im}\,\phi_v \to \pm\tan\mu$ as $x \to \pm\infty$, we conclude that the near-field solution $Z_1 = -\lambda_1\Phi_1$ matches continuously to the asymptotes (35).

Next, one of the two solvability conditions of Eq. (26) is

$$\lambda_1(\Phi_0', JZ_1) - (\Phi_0', H_1\Phi_0') = 0 .$$

Using (31), the second term vanishes and we end up with

$$\lambda_1^2(\Phi_0', J\Phi_1) = \frac{\lambda_1^2}{2}\frac{\partial P}{\partial v} = 0 . \quad (36)$$

Notice that we have written $-\lambda_1\Phi_1$ for Z_1; this is admissible since the contribution of the asymptotes (35) to the scalar product is exponentially small. If $\lambda_1 \neq 0$, we obtain $\partial P/\partial v = 0$. Otherwise, we use Eq. (31) to solve Eq. (26) explicitly: $Z_2 = \Phi_1' - \lambda_2\Phi_1$. This is again nonvanishing at infinities, but, in the first place, we know how to consistently correct the asymptotes, and in the second, the asymptotes do not contribute to the scalar product below. Using Eqs. (31) and (33) it is straightforward to check that the solvability condition for Eq. (27) is always satisfied, while the one for Eq. (28) reads

$$\lambda_2(\Phi_0', JZ_2) + \lambda_3(\Phi_0', JZ_1) - (\Phi_0', H_2 Z_0) - (\Phi_0', H_1 Z_2) = 0 .$$

Using (30) and (32), we arrive at $\lambda_2^2(\Phi_0', J\Phi_1) = 0$, whence, again, either $\lambda_2^2 = 0$ or $\partial P/\partial v = 0$. If we assume that $\lambda_2 = 0$, the condition $\partial P/\partial v = 0$ will reappear at the level $O(\epsilon^3)$, and so forth. Thus independently of the actual scaling of $\lambda(\epsilon)$, the eigenvalue may vanish only when $\partial P/\partial v = 0$—and this is exactly what we needed to prove [18].

Finally, we would like to remark that the first attempt to construct the Liapunov functional for the dark soliton was made in Ref. [19]. There are two principal distinctions between our Liapunov functional Eq. (6) and the functional suggested and numerically studied in [19]: $\tilde{\mathcal{L}} = E - \omega N - v\Pi$, where $\Pi = (i/2)\int(\psi_x^*\psi - \psi_x\psi^*)\,dx$. First, since Π is a nondifferentiable momentum, $\tilde{\mathcal{L}}$ is a nondifferentiable functional. Second, even if $\tilde{\mathcal{L}}$ had been differentiable (which is one of the defining properties of Liapunov functionals), there would have still been a range of "stable" velocities for which $\tilde{\mathcal{L}}$ is not positive definite, however. [The reason is the absence of the term $\alpha(P - P_s)^2$ from $\tilde{\mathcal{L}}$.] For instance, in the case of the dark soliton of the repulsive cubic NLS (which is of course stable for all v), the numerical analysis of [19] predicts stability only for sufficiently large velocities.

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