Universality in Sandpiles, Interface Depinning, and Earthquake Models

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Recent numerical results for a model describing dispersive transport in ricepiles are explained by mapping the model to the depinning transition of an elastic interface that is dragged at one end through a random medium. The average velocity of transport vanishes with system size L as $\langle v \rangle \sim L^{2-D} \sim L^{-0.23}$, and the avalanche size distribution exponent $\tau = 2 - 1/D \simeq 1.55$, where $D \simeq 2.23$ from interface depinning. We conjecture that the purely deterministic Burridge-Knopoff "train" model for earthquakes is in the same universality class. [S0031-9007(96)00517-0]

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Evidence for self-organized criticality (SOC) [1] has been found in controlled experiments on the granular dynamics of ricepiles [2]. By slowly adding elongated rice grains in a narrow gap between two clear plates, Frette *et al.* found that the ricepile evolves to a stationary angle of repose where on average one grain of rice falls off the edge for every grain added at the wall. Thereafter transport of the rice through the pile occurs in terms of bursts with no characteristic scale other than the system size. The rice pile exhibits SOC. Subsequently, the Oslo group has investigated tracer dispersion in the SOC pile by coloring rice grains. Christensen *et al.* [3] found that the average transport velocity of rice vanishes as the system size diverges, and correspondingly that the distribution of transit times through the pile was broad. They proposed a "sandpile" model, herein referred to as the Oslo model, to phenomenologically describe their experiments on one-dimensional ricepiles.

We establish that a broad universality class exists for SOC phenomena. The Oslo sandpile model is mapped exactly to a model for interface depinning where the interface is slowly pulled at one end through a medium with quenched random pinning forces. The height of the interface maps to the number of toppling events in the sandpile model. The annealed noise of the random thresholds for toppling sand grains maps to quenched pinning forces for the interface. Thus a problem of dispersive transport [4] in a granular medium can be recast in terms of the somewhat better understood problem of interface depinning [5]. This leads to a number of scaling relations expressing critical exponents in the Oslo sandpile model in terms of the avalanche dimension D. This quantity is equal to the avalanche dimension for a uniformly driven interface, which has been determined numerically to be $D = 2.23 \pm 0.03$ [5]. We also predict that the avalanche size distribution exponent $\tau = 2 - 1/D \simeq 1.55$ in the Oslo model, and that the average velocity of transport vanishes as $\langle v \rangle \sim L^{2-D} \sim L^{-0.23}$ when time is measured in units of the number of sand grains added. These predictions agree with previous numerical simulation results [3], and with our simulation results. Finally, we conjecture that the Burridge-Knopoff [6] train model studied by de Sousa Vieira [7], a purely deterministic mechanical model with no embedded randomness of a block-spring chain pulled at one end, is also in the same universality class.

The Oslo sandpile model is defined as follows: In a one-dimensional system of size L, an integer variable $h(x)$ gives the height of the pile at position x, and $z(x) = h(x) - h(x+1)$ is the local slope. The boundary condition is $h(L+1) = 0$. Grains are dropped at $x = 1$ until the slope $z(1) > z^c(1)$; then the site topples and one grain is transferred to the neighboring site on the right $x = 2$. At each subsequent time step, all sites x with $z(x) > z^c(x)$ topple in parallel. In a toppling event at site x, $h(x) \to h(x) - 1$ and $h(x+1) \to h(x+1) + 1$. No grains are added to the pile until the avalanche resulting from adding a sand grain ends and the system reaches a stable state with $z(x) \leq z^c(x)$ for all x. The key ingredient making this model different from previous sandpile models [1,8] is that the critical slopes $z^c(x)$ are dynamical variables chosen randomly to be 1 or 2 every time a site topples. The annealed randomness describes in a simple way the changes in the local slopes observed in the ricepile experiments [3].

It is useful to define a local force

$$F(x,t) = h(x,t) - h(x+1,t) - \eta(x,H), \quad 1 \leq x \leq L, \tag{1}$$

where η are the randomly distributed critical slopes, i.e., $\eta(x,H) = z^c(x)$ which take integer values 1 or 2 with equal probability. The boundary condition is $h(L+1,t) = 0$ for all times. At each time step $t \to t+1$, all unstable sites where $F(x,t) > 0$ topple. The quantity $H(x,t)$ in Eq. (1) is the total number of toppling events at site x up to time t. The threshold slope at a site is chosen randomly after each toppling event at that site; hence $\eta(x,H)$ is an uncorrelated quenched random variable in the space of (x,H). The dynamics is central seeding; when all sites have reached a stable state where $F(x) \leq 0$, a grain of sand is added at site 1, $h(1) \rightarrow h(1) + 1$, $t \rightarrow t + 1$, and a new avalanche starts. It is easy to see that the sandpile dynamics of toppling events is traced out by an advancing interface where the height profile of the interface, $H(x,t)$, is the accumulated number of topplings.

When the initial condition is an empty sandpile, $h(x, t = 0) = 0$ for all x, and the dynamics is centrally seeded, the number of sand grains at x at time t is the local gradient in the number of topplings that have occurred up to that time,

$$h(x,t) = H(x - 1, t) - H(x,t). \qquad (2)$$

As a result, Eq. (1) can be rewritten as a dynamical equation for an interface with height profile $H(x,t)$,

$$F(x,t) = \nabla^2 H(x,t) - \eta(x,H), \qquad 1 \leq x \leq L, \quad (3)$$

where the discretized Laplacian $\nabla^2 H(x) = H(x - 1) - 2H(x) + H(x + 1)$. The interface dynamics is that for all x where $F(x,t) > 0$, $H(x, t + 1) = H(x,t) + 1$ and the site advances; otherwise $H(x, t + 1) = H(x,t)$ and the site is pinned. The boundary condition is $H(L + 1, t) = H(L,t)$ for all t. Whenever the interface becomes stuck so that $F(x) \leq 0$ for all x, it is pulled at the boundary at the origin, $H(0, t + 1) = H(0,t) + 1$. This is an example of depinning of an elastic interface which has been widely studied [5,9,10]. The difference here is that rather than being driven uniformly the interface is driven by being slowly dragged at the boundary. After a sufficient amount of motion has occurred, the interface approaches a self-organized critical depinning transition. In the critical state, information about the pull at one end can be communicated throughout the entire length of the interface. This occurs when the interface finds an average curvature which precisely balances the pinning forces. This corresponds to the sandpile reaching its critical angle of repose, where it can transfer sand out from the origin to the edge of the pile.

In order to proceed, we briefly review some known results for interface depinning with uniform driving [5]. The depinning transition can be reached either by applying a constant force or constant velocity constraint. In constant force depinning a uniform force is applied to all sites, i.e., add a term F_{ext} to the right hand side of Eq. (3), and advance all unstable sites with $F > 0$ in parallel. When F_{ext} is tuned to F_c a depinning transition occurs. The constant velocity depinning transition is an attractor for an extremal dynamics where the unique site along the interface with the largest force $F(x,t)$ is advanced. Now the dynamics occurs in series with one site advancing at each step rather than in parallel. The extremal model self-organizes to the critical state of constant velocity depinning. As explained in Ref. [5], the critical exponents relating to the physical extents of complete avalanches are the same in the constant force and constant velocity cases, i.e., the roughness exponent χ, the avalanche dimension D, the exponent for the distribution of avalanche sizes τ. Since the dynamics is different, though, the critical exponents referring to propagating avalanches are different, i.e., the dynamical exponent z, the fractal dimension of unstable sites d_f, and the average growth of activity η. Note that the boundary driven interface combines certain aspects of these two cases for uniform driving. The dynamics occurs in parallel with all unstable sites advancing as for constant force depinning. The criticality is self-organized as in the extremal constant velocity case.

The size of an avalanche, s, for the interface is the integrated area during the burst resulting from pulling the end once. It is the difference between the final height profile after the pull and the initial one before the pull. In the Oslo model, s is the total number of toppling events which occur after adding one grain of sand at the origin. As for the case of uniform driving, the distribution of avalanche sizes is observed numerically to obey a scaling form [3]

$$P(s) \sim s^{-\tau} G(s/L^D), \qquad (4)$$

which is a power law with a cutoff that grows with system size $s_{\text{co}} \sim L^D$, where D is the avalanche dimension. We now argue that D for the boundary driven interface, and hence for the Oslo sandpile model, is the same as for the uniformly driven interface. The amount of motion required to reach the depinning transition starting from an arbitrary configuration scales as L^D since it involves a system wide avalanche. During the transient approach to the depinning transition, the correlation length is less than the system size L. Depinning occurs precisely when every site has moved at least once. At this instant, but not before, it is possible to communicate information from the boundaries throughout the one-dimensional system. Thus the scaling of the amount of motion for the SOC attractor to be reached is independent of the boundary condition, i.e., whether we have boundary driven SOC or extremal, uniformly driven SOC. It depends only on the anomalous diffusive dynamics [5,11] of avalanches. This hypothesis is confirmed by numerical simulations of the Oslo model giving $D = 2.25 \pm 0.10$ [3] and extremal interface depinning giving $D = 2.23 \pm 0.03$ [5].

Since the avalanche is a compact object, the size of an avalanche $s \sim r r_\perp$, where r is the spatial extent of the avalanche along the internal interface coordinate and $r_\perp$ is the maximum extension in the direction of growth. If there is only one length scale for motion in the direction of growth, then $r_\perp \sim r^\chi$, with the roughness exponent defined by the divergence of interfacial height fluctuations with system size $w(L) \sim L^\chi$. We numerically measured $H = r_\perp$, the maximum number of topplings at a site vs r for 10^7 avalanches in the Oslo model and found $H \sim r^{1.23 \pm 0.03}$, in excellent agreement with our prediction $\chi = D - 1$. Since the height of the sandpile $h \sim dH/dx$, the roughness exponent of the surface of the sandpile model is $\chi_{\text{pile}} = \chi - 1 = D - 2 \simeq 0.23$.

The distribution of avalanche sizes for the Oslo model, however, is different from the uniformly driven interface due to a conversion law. From Eq. (4), the average size of an avalanche diverges with the system size as $\langle s \rangle \sim L^{\lambda/\nu}$ where $\gamma = \nu D(2 - \tau)$. Since on average each site must topple exactly once in the critical state in order to transport the added grain of sand to the boundary,

$$\langle s \rangle = L \quad \text{and} \quad \tau = 2 - 1/D\,. \tag{5}$$

Using $D = 2.23$, we predict $\tau = 1.55$ in precise agreement with the measured value $\tau = 1.55 \pm 0.10$ [3] for the Oslo model. For the boundary driven interface, this conservation law expresses the constraint that on average each site must advance one step when the end is dragged one step. The value $\tau \simeq 1.55$ for the boundary driven interface is far from $\tau \simeq 1.13$ [5] measured when the interface is driven uniformly either at constant force or constant velocity. These latter systems do not obey Eq. (5).

Christensen *et al.* [3] measured the average velocity $\langle v \rangle$ of tracer grains in transit through the pile. It was found to vanish as $\langle v \rangle \sim L^{-0.3\pm0.1}$ in the model. The time unit used to measure velocity was the number of sand grains added. Based on another conservation law, it was argued that the average velocity in this time unit scales as the inverse width of the height fluctuations of the sandpile, i.e., $\langle v \rangle \sim 1/w_{\mathrm{pile}}(L) \sim L^{-\chi_{\mathrm{pile}}}$. This occurs because the model completely separates into a frozen bulk phase where the grains never move and an active surface zone of width w_{pile} where transport takes place. The collection of grains in the active zone moves on average as an incompressible object upon addition of a sand grain. From our result $\chi_{\mathrm{pile}} = D - 2$, we predict $\langle v \rangle \sim L^{2-D} \sim L^{-0.23}$, in reasonable agreement with the numerical simulation results.

It is important to notice that the time unit for the Oslo model is different from the usual time unit used for interface depinning since each sand grain dropped results in many update steps, equal on average to the average duration of an avalanche $\langle t \rangle$. In accordance with Eq. (4), we propose that the distribution of avalanche durations, t, is given by

$$P(t) \sim t^{-\tau_t} G(t/L^z)\,, \tag{6}$$

where the cutoff in avalanche durations $t_{\mathrm{co}} \sim L^z$, and from conservation of probability $z(\tau_t - 1) = D(\tau - 1)$. Thus, from Eq. (6) $\langle t \rangle$ is diverging in the thermodynamic limit $L \to \infty$ as $\langle t \rangle \sim L^{z(2-\tau_t)} \sim L^{D(1-\tau)+z} \sim L^{z+1-D}$. Measuring time in the simulation in terms of update steps, t, rather than sand added, we find $\langle v \rangle \sim L^{1-z} \sim L^{-0.42}$. We numerically measured the duration of avalanches, t vs r where $t \sim r^z$, for 10^7 avalanches in the Oslo model and found $z = 1.42 \pm 0.03$ which is the same as Leschhorn's numerically measured value $z \simeq 1.42$ for the constant force depinning transition [10]. Since the average duration of avalanches is diverging with L, sand must be added slower and slower as the system size is increased in order to stay at criticality.

Burridge and Knopoff [6] introduced a mechanical model for the stick-slip dynamics of earthquake faults. It consists of blocks connected by harmonic springs sliding with friction. The first element of the block-spring chain is connected to a driver that moves at constant velocity. It is referred to as the train model [7]. de Sousa Vieira found that the train model exhibits SOC unlike some other block-spring systems [12] where every element is connected to the driver. The train model is completely deterministic and contains *no quenched randomness nor randomness in the initial conditions.* The equation of motion for the position of the jth block, U_j, is

$$\ddot{U}_j = U_{j+1} - 2U_j + U_{j-1} - \Phi\left(\frac{\dot{U}_j}{\nu_c}\right). \tag{7}$$

The static friction force $\Phi(0) = 1$, which is weakened at finite velocity

$$\Phi(\dot{U}/\nu_c) = \frac{\mathrm{sgn}(\dot{U})}{1 + \dot{U}/\nu_c}\,. \tag{8}$$

The equation of motion is valid if the sum of elastic forces is greater than the static friction force, otherwise $\dot{U}_j = 0$. The train model has precisely one positive Lyapunov exponent giving chaotic behavior [13]. The blocks in this system exhibit stick-slip dynamics with a power law distribution of event sizes and extents. A sum rule for the moments of slipping events corresponding to Eq. (5) holds since on average every block must move with an average velocity equal to the pulling speed, but this motion takes place intermittently in terms of bursts [7].

We conjecture that the train model is in the same universality class as the Oslo model and boundary driven interface depinning. Since the model is dissipative and exhibits SOC, it is reasonable that a dissipative term $\dot{U}$ would dominate the acceleration term in Eq. (7) at long length and time scales. After a slip event, the blocks come to rest in a new configuration with a random elastic force increment at each site required to induce a subsequent slip event or toppling. This is the result of the chaotic dynamics, and in a coarse grained picture can be described by quenched random thresholds for static friction $\Phi(0)$ in the space of position and events, which corresponds to $\eta(x, H)$. Such an equivalence of a deterministic model with no embedded randomness which is chaotic with a stochastic model also occurs between the deterministic Kuramoto-Shivashinsky [14] equation and the Langevin equation proposed by Kardar, Parisi, and Zhang [15]. Indeed, numerical simulations of the train model give $\tau - 1 \simeq 0.6$ and $\tau_R = 1 + D(\tau - 1) = D \simeq 2.2$ [7], which agree with the critical exponents measured for the Oslo model and support our conjecture.

Our results imply that broad universality classes in self-organized critical phenomena exist. In particular we establish that the Oslo sandpile model for transport in granular piles is in the same universality classes as the depinning transition of an interface when it is slowly dragged at one end through a random medium. The mapping unifies the mechanism giving dispersive transport with long-tailed waiting time distributions in granular media with avalanche dynamics of interface depinning. The purely deterministic train model of Burridge and Knopoff [6] describing earthquakes is proposed to be in the same universality class.

While this manuscript was in preparation, we became aware of a work by Cule and Hwa [16] who demonstrate that a similar block-spring model with quenched random spring constants and purely dissipative dynamics is in the universality class of interface depinning with uniform driving.

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[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. A **38**, 364 (1988).

[2] V. Frette, K. Christensen, A. Malthe-Sørenssen, J. Feder, T. Jøssang, and P. Meakin, Nature (London) **379**, 49 (1996).

[3] K. Christensen, A. Corral, V. Frette, J. Feder, and T. Jøssang, preceding Letter, Phys. Rev. Lett. **77**, 107 (1996).

[4] H. Scher, M. Shlesinger, and J. Bendler, Phys. Today **44**, No. 1, 26 (1991).

[5] M. Paczuski, S. Maslov, and P. Bak, Phys. Rev. E **53**, 414 (1996).

[6] R. Burridge and L. Knopoff, Bull. Seismol. Soc. Am. **57**, 341 (1967).

[7] M. de Sousa Vieira, Phys. Rev. A **46**, 6288 (1992).

[8] L. Kadanoff, S. Nagel, L. Wu, and S. Zhou, Phys. Rev. A **39**, 6524 (1989).

[9] M. V. Feigelman, Sov. Phys. JETP **58**, 1076 (1983); R. Bruinsma and G. Aeppli, Phys. Rev. Lett. **52**, 1547 (1984).

[10] H. Leschhorn, Physica (Amsterdam) **195A**, 324 (1993).

[11] M. Paczuski and S. Boettcher, Report No. COND-MAT 9603085.

[12] J. M. Carlson and J. S. Langer, Phys. Rev. Lett. **62**, 2632 (1989); J. M. Carlson, J. S. Langer, and B. E. Shaw, Rev. Mod. Phys. **66**, 657 (1994).

[13] M. de Sousa Vieira and A. J. Lichtenberg, Phys. Rev. E **53**, 1441 (1996).

[14] V. Yakhot, Phys. Rev. A **24**, 642 (1981).

[15] M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986); for a review, see T. Halpin-Healy and Y.-C. Zhang, Phys. Rep. **254**, 215 (1995); J. Li and L. M. Sander, Fractals **3**, 507 (1995).

[16] D. Cule and T. Hwa, Phys. Rev. Lett. (to be published).