Diffusion due to Beam-Beam Interaction and Fluctuating Fields in Hadron Colliders

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(Received 14 February 1996)

Random fluctuations in the tune, beam offsets, and beam size in the presence of the beam-beam interaction are shown to lead to significant particle diffusion and emittance growth in hadron colliders. We find that far from resonances high frequency noise causes the most diffusion while near resonances low frequency noise is responsible for the large emittance growth observed. Comparison of different fluctuations shows that offset fluctuations between the beams cause the largest diffusion for particles in the beam core. [S0031-9007(96)00857-5]

PACS numbers: 41.75.−i, 29.20.Dh, 29.27.Bd

Emittance growth due to the nonlinear beam-beam interaction is a major concern at all hadron colliders. The tunes (i.e., the rotation numbers of the transverse betatron oscillations) are always chosen to avoid low-order resonances and analysis shows that particle amplitude growth under typical static conditions is not significant. However, deterministic and stochastic variations of parameters in time change the dynamics qualitatively. The effects of deterministic tune modulation are well studied, and removing modulation lines from the betatron spectrum reduces particle loss from the tails of the beam [1]. Random fluctuations in the tune, closed orbit, and beam size are also present in accelerators. Qualitative arguments [2] and numerical simulations [3] have shown that tune fluctuations lead to emittance growth especially for tunes close to a resonance. Although experiments have been done to measure the diffusion rate of large-amplitude particles [4–6], no comparable measurements exist yet under luminosity conditions for small-amplitude particles in the beam core. This is the region where the beam-beam interaction is dominant and where our theory is expected to explain quantitatively the amplitude dependence of diffusion.

As we shall discuss below, the one-dimensional expressions for the off-resonance diffusion coefficients contain qualitatively the same physics as the more realistic two-dimensional expressions. Only near resonance does the two-dimensional case have new features. All accelerators are operated far from low-order resonances, so it is valid to draw on the one-dimensional results to explain the underlying physics. We consider collisions of a proton beam with an opposing beam composed of either leptons as at HERA or hadrons as at the Fermilab Tevatron and the proposed LHC at CERN. For one-dimensional motion, the beam-beam potential seen by a proton, assuming a Gaussian charge distribution of the opposing beam, is given by $U(x) = C\int_0^\infty dq\,[1 - e^{-x^2/(2\sigma_{\rm op}^2+q)}]/(2\sigma_{\rm op}^2 + q)$. The constant is $C = N_{b,\rm op}r_p/\gamma_p$ where $N_{b,\rm op}$ is the number of particles per bunch in the opposing beam, r_p is the classical radius of the proton, γ_p is the relativistic kinematic factor for the protons, and $\sigma_{\rm op}$ is the rms size of the opposing beam. Transforming to action-angle coordinates (J,ψ) via $x = \sqrt{2J\beta^*}\cos\psi$, $x' = -\sqrt{2J/\beta^*}\sin\psi$ where β^* is the β function at the interaction point, we obtain the Fourier expansion of the potential

$$U(x) = C\sum_{k=0}^{\infty} U_k(a)\cos 2k\psi\,, \tag{1}$$

where $a = \beta^* J/2\sigma_{\rm op}^2$ is a dimensionless amplitude. The Fourier amplitudes are

$$U_k = \int_0^a \frac{1}{w}\,[\delta_{0k} - (2 - \delta_{0k})(-1)^k e^{-w} I_k(w)]\,dw\,, \tag{2}$$

where the I_k are modified Bessel functions. Including the linear motion and the beam-beam interaction, the Hamiltonian is $H = \nu^0 J + U(J,\psi)\delta_p(\theta)$ where ν^0 is the nominal tune, $\delta_p(\theta)$ is the periodic delta function with period $2\pi/N_{\rm IP}$, $N_{\rm IP}$ being the number of interaction points, and θ, the "time" variable, advances by 2π per turn. Integrating the equations of motion over one turn leads to the one-turn beam-beam map: $\Delta\psi = 2\pi\nu^0 + \partial U/\partial J$ and $\Delta J = -\partial U/\partial\psi$.

First we consider the diffusion in amplitude due to random fluctuations in the tune. Usually the random contribution to the tune is quite small, of the order of 0.001 at the most, but this is sufficient to affect the long time dynamics. The sources of tune fluctuation include power supply noise in quadrupoles, closed orbit fluctuations through the nonlinear magnets and mechanical vibrations of the nonlinear magnets. In addition, scattering with the residual gas molecules, intrabeam scattering due to the Coulomb force, and RF noise lead to fluctuating particle momenta. This in turn leads to a tune fluctuation via the machine chromaticity. We model the tune fluctuation by an additional term $\Delta\psi_r$ in the total phase. Assuming that the random contribution is small, we can write the change in action at turn m due to this fluctuating phase alone as $\Delta J_r(m) =$

$[d\Delta J(m)/d\psi]\Delta\psi_r(m) + O(\Delta\psi_r^2)$. The unperturbed total phase at turn m and action J is $\psi(m) = 2m\pi\nu(J) + \psi_0$, where $\nu(J) = \nu^0 + \Delta\nu(J)$ is the tune including the beam-beam induced tune shift $\Delta\nu(J)$ and ψ_0 is the initial phase. When the tune is far from a resonance, the linear action J is conserved after averaging. This allows us to assume that $\Delta J_r(m) \ll J(0)$ so that in the sum over turns we can replace $J(m)$ by $J(0)$. We assume that the random process is stationary so that the random phase correlation function is of the form $\langle\Delta\psi_r(l)\Delta\psi_r(n + l)\rangle = 4\pi^2\Delta\nu_r^2 K_\nu(n)$, where the average is over many noise realizations, $\Delta\nu_r$ is the amplitude of the tune fluctuations, and $K_\nu(-n) = K_\nu(n)$. The diffusion coefficient defined as $D_\nu(J) \equiv \lim_{N\to\infty}\langle[J(N) - J(0)]^2\rangle/N$ is found, by extracting the dominant terms, to be

$$D_\nu(J) = 32(\pi C\Delta\nu_r)^2 \sum_{k=1}^{\infty} k^4 U_k^2 \sum_{n=-\infty}^{\infty} K_\nu \cos 4\pi k\nu n\,. \tag{3}$$

We observe that *far from resonances only the tune noise at even harmonics of the betatron tune leads to a diffusion in the action.* A natural choice to model the tune fluctuations is the Ornstein-Uhlenbeck (OU) process because it is the only Gaussian stationary Markov process and the spectral density $S(\omega)$ (related to K_ν by the cosine transform) decays as ω^{-2} which is in reasonable agreement with measured noise densities. For the discrete time OU process with correlation time τ_c, the correlation function is $K_\nu(n) = (1 - 1/\tau_c)^{|n|}/[1 - 1/2\tau_c]$. The spectral density drops to roughly half its maximum value at a frequency $f_{1/2} = f_{\rm rev}/2\pi\tau_c$. The revolution frequency $f_{\rm rev}$ at HERA is 47.3 kHz. Substituting this form for K_ν leads to

$$D_\nu(J) = 32\,\frac{(\pi C\Delta\nu_r)^2}{1 - 1/2\tau_c} \sum_{k=1}^{\infty} \frac{k^4 U_k^2 \sinh\Theta}{\cosh\Theta - \cos 4\pi k\nu}\,, \tag{4}$$

where $\Theta = -\ln(1 - 1/\tau_c)$. The main amplitude dependence of $D_\nu(J)$ is contained in the Fourier coefficients. From the expansion of the dominant coefficient U_1 we find that at small amplitudes $D_\nu(J) \sim J^2$. At large amplitudes the beam-beam force vanishes and all Fourier coefficients U_k go to constant values. Hence the diffusion coefficient $D_\nu(J)$ due to the beam-beam interaction increases monotonically in the core of the beam and levels off at large amplitudes. Magnetic multipole nonlinearities, not included in our analysis, contribute significantly to the transport of particles only in the tails of the beam. This results in the diffusion coefficient increasing with amplitude even at large amplitudes.

We have compared the above analysis with a numerical calculation. An initial distribution of 1000 particles is placed at 100 different amplitudes with 10 particles at each amplitude distributed uniformly in phase. The particles are tracked for 10^7 turns or more (the number increasing with the noise correlation time) using the beam-beam map with tune fluctuations. The diffusion coefficients at each amplitude are averaged over the ten phases and over ten noise realizations. The simulations were done for three correlation times: $\tau_c = 1.1$, 10, 100. Figure 1 shows a comparison of the diffusion coefficient obtained from Eq. (4) with that calculated numerically at $\tau_c = 10$. This good agreement holds for the other correlation times as well. Both the analysis and the numerical results show that $D_\nu(J) \sim \tau_c^{-1}$ for large τ_c. High frequency noise (above 1 kHz, say) is strongly attenuated within the interior of the beam pipes by eddy currents within the metallic liner of the beam pipes and also by the impedance of the magnets and other devices in a storage ring so realistic values of the noise correlation time for HERA are $\tau_c \geq 10$.

The diffusion in amplitude causes emittance growth over the period of stored beam—typically 24 to 30 h for the proton beam at HERA. The emittance evolution can be followed by solving the Fokker-Planck equation. Assuming that the diffusion in action is a Markov process and the drift coefficient is half the derivative of the diffusion coefficient (as is usual for a Hamiltonian system) [7], the beam phase-space density ρ evolves according to the Fokker-Planck equation,

$$\frac{\partial\rho}{\partial t} = \frac{1}{2}\frac{\partial}{\partial J}\left(D(J)\,\frac{\partial\rho}{\partial J}\right). \tag{5}$$

We integrate this one-dimensional Fokker-Planck equation by the method of lines [8]. An absorbing boundary is placed at an action J_b corresponding to the position of the beam pipe. The density at the origin does not change since the diffusion coefficient and its derivative vanish there. The evolution of the average action is then found from $\langle J(t)\rangle = \int_0^{J_b} J\rho(J,t)\,dJ / \int_0^{J_b} \rho(J,t)\,dJ$. Figure 2 shows the evolution of the average action for three correlation times. For noise with the highest frequency content, $\tau_c = 1.1$, $\langle J\rangle$ grows the most rapidly as expected, then decreases as particles are lost at the boundary. Diffusion is slower

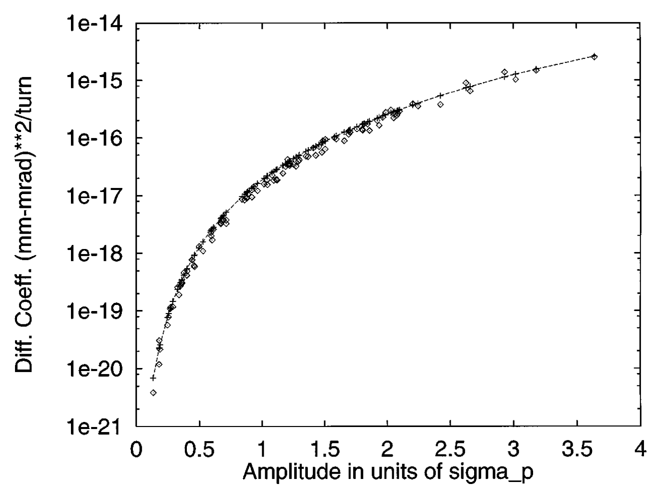

FIG. 1. The diffusion coefficient $D_\nu(J)$ calculated analytically from Eq. (4) (dashed line) compared with the values obtained from the simulation. σ_p is the rms size of the proton beam. Parameter values $\nu^0 = 0.291$, $\Delta\nu_r = 10^{-4}$, $\tau_c = 10$, $N_{b,\rm op} = 3.8 \times 10^{10}$, $\gamma_p = 874$, $\beta^* = 7.0$ m, and $\sigma_{\rm op} = 0.286$ mm.

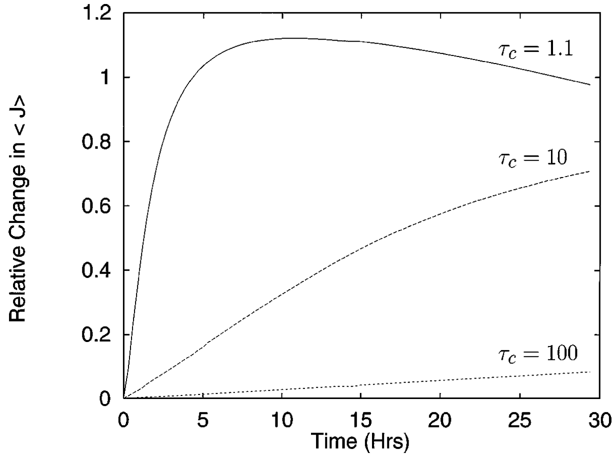

FIG. 2. Relative growth in the average action $\langle J \rangle$ of a proton beam over a storage time of 30 h due to tune fluctuations at three noise correlation times. Parameters are the same as in Fig. 1.

for the other two correlation times and particles are not lost at the boundary so $\langle J \rangle$ grows almost linearly with time. These calculations show that even in this one dimensional model tune fluctuation can cause the emittance to grow by 10% to 70% over the storage time in a proton machine.

Emittance growth due to fluctuations is significantly enhanced near a resonance. As the tunes approach a resonance, the resonance islands increase in width and the resonant amplitudes move further out. Finally, at the resonance tunes the fixed points are at infinity because the beam-beam tune shift is largest at the origin and vanishes at infinity. The phase-space portraits at $\nu^0 = 0.25$, for example, are diamond shaped close to the origin, and at large amplitudes they are four-armed stars with long arms along the four axes. Particle motion with external fluctuations has two aspects: motion on a level curve, labeled by a transverse energy E, and diffusive motion between level curves. Near resonance a small-amplitude particle may be diffusively transported to a resonance island where it experiences a large jump in amplitude. Exactly on resonance, e.g., at $\nu^0 = 0.25$, a particle may, after a long time, diffuse on to a star-shaped curve which subsequently leads to a very large-amplitude excursion.

To analyze the diffusive motion we observe that without noise and even after averaging, the linear invariant J is not conserved near resonance. Instead, after averaging over the fast varying phases, a time-independent Hamiltonian is obtained which describes motion close to the $2k$th integer resonance (tune $\nu_{2k} = \text{integer}/2k$),

$$H_{2k} = \delta J + A \sum_{m=0}^{\infty} U_{mk} \cos 2mk\phi \,, \qquad (6)$$

where $\delta = \nu^0 - \nu_{2k}$, the difference from the resonance tune, $\phi = \psi - \nu_{2k}\theta$, the slowly varying phase, and $A = C/2\pi$. J oscillates periodically betweeen two limits $J_{\min}$ and $J_{\max}$ which are determined by the transverse energy $E = H_{2k}$. Dropping all U_{mk}, $m > 1$, the period on a curve labeled by E is $T_E = (1/2\pi k) \times \int_{J_{\min}(E)}^{J_{\max}(E)} dJ/\sqrt{A^2 U_k^2 - (E - \delta J - AU_0)^2}$. In contrast to the betatron tune ν, this tune $\nu_E = 1/T_E$ is very small, typically of the order of 0.001 for $k = 2$ and $\delta \sim 10^{-4}$. ν_E increases with the resonance order $2k$. The tune fluctuations $\Delta\nu_r$ cause the resonant amplitude and the island widths to also fluctuate and the energy to diffuse. The rate of change of H_{2k} is found to be $dH_{2k}/d\theta = -(dJ/d\theta)_u \Delta\nu_r$, the subscript u denotes the unperturbed rate of change. After integrating over a turn, the total change in the Hamiltonian at turn N is $H_{2k}(N) - H_{2k}(0) = -\sum_{m=1}^{N}[J(m) - J(m-1)]\Delta\nu_r(m)$. Expanding J in a Fourier series, $J(m) = \sum_{j=0}^{\infty} B_j \cos(2\pi\nu_E jm + \theta_j)$ and using the stationarity of $\langle \Delta\nu_r(l)\Delta\nu_r(l+n)\rangle$, we obtain for the diffusion of the energy

$$D_\nu(E) = \frac{(\Delta\nu_r)^2}{1 - 1/(2\tau_c)} \sum_{j=1}^{\infty} \frac{B_j^2(1 - \cos 2\pi j\nu_E)\sinh\Theta}{\cosh\Theta - \cos 2\pi j\nu_E} \,. \qquad (7)$$

Diffusion at a given tune increases smoothly moving out from the origin, jumps when the particle is on the largest of the resonant islands, decreases to zero at the stable fixed point, increases back to the value on the largest island, and stays nearly constant thereafter. The noise frequencies which contribute to the diffusion in energy are the harmonics of the low frequency $\nu_E f_{\text{rev}}$. *The topology of the phase-space orbits and the fact that noise of comparatively low frequencies has the dominant contribution to the diffusion in energy explains the large growth in emittance due to noise in the neighborhood of a resonance.*

Next we consider fluctuations of the offset between the beams at the interaction point (IP). The position of the maximum of the beam-beam force fluctuates so more particles in the proton beam will be subjected to a larger force. It also destroys the symmetry of the beam-beam force and can excite odd order resonances. Our results below for the general potential of Eq. (1) generalizes the earlier result of Stupakov [9] for a flat beam. We assume that the offset fluctuation $d_r(m)$ at turn m is small and write it as $d_r(m) = \Delta d_r \chi(m)\sigma_{\text{op}}$ where Δd_r is the dimensionless amplitude of the offset and $\chi(m)$ is a random variable of zero mean and unit variance. Calculation of the diffusion coefficient far from resonances yields

$$D_{\text{off}}(J) = \frac{1}{8}(C\sigma_{\text{op}}\Delta d_r)^2 \sum_{k=0}^{\infty}(2k+1)^2 G_k^2(a) \times \sum_{n=-\infty}^{\infty} K_{\text{off}}(n)\cos 2\pi(2k+1)\nu n \,. \qquad (8)$$

The correlation function is $K_{\text{off}} = \langle \chi(l)\chi(n+l)\rangle$. G_k, the Fourier coefficients of the beam-beam force, are given by $G_k = \sqrt{a}[U'_{k+1} + U'_k]/\sigma_{\text{op}} + [(k+1)U_{k+1} - kU_k]/\sqrt{a}\,\sigma_{\text{op}}$. Notice here that the odd harmonics of the betatron tune contribute to the diffusion in action.

When the size of the opposing beam fluctuates, both the location of the maximum of the beam-beam force ($\propto \sigma_{\mathrm{op}}$) and the maximum ($\propto \sigma_{\mathrm{op}}^{-1}$) also fluctuate. Consequently, protons in a larger range of amplitudes are subject to the maximum of the force—as with offset fluctuations. A study of this for the flat beam potential was reported recently in [10]. We find that the diffusion coefficient for the general beam-beam potential is

$$D_{\sigma_{\mathrm{op}}}(J) = 8(C\Delta\sigma_r a)^2 \sum_{k=1}^{\infty}[kU_k']^2 \sum_{n=-\infty}^{\infty} K_{\sigma_{\mathrm{op}}} \cos 4\pi k\nu n \,. \tag{9}$$

The size fluctuation at turn n is $\Delta\sigma_r \eta(n)\sigma_{\mathrm{op}}$ where $\Delta\sigma_r$ is dimensionless, $\eta(n)$ is a random variable of mean zero and unit variance, and $K_{\sigma_{\mathrm{op}}}(n) = \langle \eta(l)\eta(n+l)\rangle$.

Figure 3 compares the analytical diffusion coefficients from the three fluctuating phenomena considered here. We find that diffusion due to offset fluctuations is largest for amplitudes up to twice the rms proton beam size. At greater amplitudes, diffusion due to beam size and offset fluctuations, both of which directly affect the amplitude of the beam-beam kick, are of the same order of magnitude [11]. Diffusion due to tune fluctuations is the smallest at all amplitudes because it affects only the phase at which the particle is kicked. Nevertheless, the sources of tune fluctuations are difficult to eliminate and more numerous than for the other fluctuations.

Off resonances, the above formulas for the diffusion coefficients can be extended to 2 degrees of freedom in a straightforward fashion [12] and lead to similar conclusions. Near resonances, the significant difference is that diffusion is enhanced only close to low-order resonances for the one-dimensional case, but for the two-dimensional interaction even relatively high-order resonances, e.g., 14th order can lead to large emittance growth, as was also observed in [3]. A detailed study of the two-dimensional case will appear separately [12]. A comparison with experiment requires that diffusion coefficients be measured over a wide range of amplitudes ranging from the beam core to the tails. To date, the measurements reported so far [4–6] have been for a limited range of amplitudes within the beam halo. Measurements of diffusion coefficients within the beam core will require new techniques such as one with beam echoes recently used to measure longitudinal diffusion rates [13].

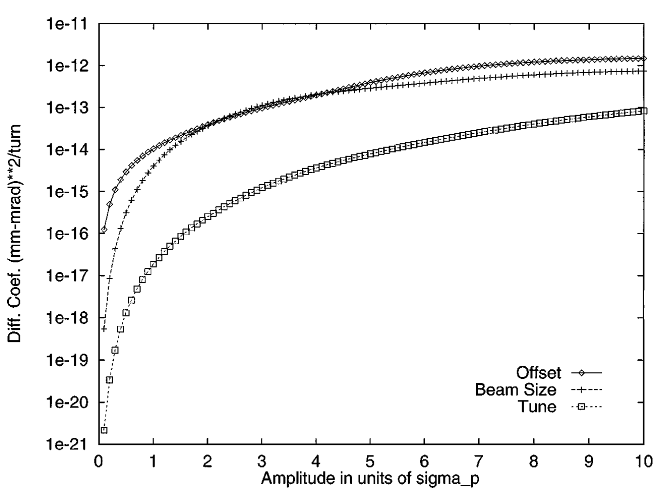

FIG. 3. Comparison of the diffusion coefficients in J due to tune fluctuations, offset fluctuations, and beam size fluctuations given by Eqs. (4), (8), and (9), respectively. $\tau_c = 10.0$, $\Delta\nu_r = 10^{-4}$, and $\Delta d_r = 0.01 = \Delta\sigma_r$. Other parameters are the same as in Fig. 1.

We summarize the three main results in this Letter. Far from low-order resonances, high frequency tune fluctuations cause larger growth of particle amplitudes than low frequency fluctuations. These high frequency fluctuations can cause the emittance to nearly double over the storage time of a day. Near resonances, low frequency fluctuations are resonant with the motion of the linear invariant, and these lead to the largest diffusion in the energy which subsequently leads to significant emittance growth. Comparing different fluctuations in the off-resonance case, we have found that for reasonable values of the fluctuating amplitudes, offset fluctuations at the IPs cause the largest diffusion at small amplitudes, while at large amplitudes, fluctuations in the size of the opposing beam have a comparable effect as the offset fluctuations.

We thank A. Bazzani, H. Mais, and F. Willeke for very fruitful discussions and support.

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