Zero-Brane Quantum Mechanics

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We consider low energy, nonrelativistic scattering of two Dirichlet zero-branes as an exercise in quantum mechanics. For weak string coupling and sufficiently small velocity, the dynamics is governed by an effective U(2) gauge theory in $0 + 1$ dimensions. At low energies, D-brane scattering can reliably probe distances much shorter than the string scale. The only length scale in the quantum mechanics problem is the eleven-dimensional Planck length. This provides evidence for the role of scales shorter than the string length in the weakly coupled dynamics of type IIA strings. [S0031-9007(96)00710-7]

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Dirichlet branes (D-branes) [1–3] in string theory have been intensively studied since the realization that they carry Ramond-Ramond charge and are responsible for many of the nonperturbative effects required by duality [4–16]; for a recent review, see [17]. At energies below the string scale, the dynamics of D-branes should be governed by the effective theory of the massless modes of the open strings which end on the branes. For a single p-brane, this effective theory is the reduction of a ten-dimensional $N = 1$ supersymmetric Abelian gauge theory to the $p + 1$ dimensional world volume of the brane. The $p + 1$ components of the gauge field tangent to the brane give rise to an internal gauge theory on the world volume, while the $9 - p$ scalars which arise from components of the gauge field normal to the brane are interpreted as position coordinates that translate the brane [1, 2]. When n-branes are present, open strings carry a Chan-Paton factor to label which brane they end on, and the effective gauge theory gets enlarged to a U(n) gauge group. This description has been used to analyze bound states of D-branes [5].

We consider Dirichlet zero-branes, or D-particles, in ten-dimensional type IIA string theory. The IIA string can be obtained by compactifying M theory on a circle [18, 19], and the low energy effective theory of strongly coupled type IIA string theory is known to be eleven-dimensional supergravity. The zero-branes we study have an interpretation as Kaluza-Klein modes of the eleven-dimensional theory. This suggests that eleven-dimensional physics should play a role in their dynamics; we will indeed find evidence for this below.

One of the more intriguing hints from duality is the appearance of new length scales in string theory [20]. In ten dimensions, the IIA string is characterized by the string length $l_s = 1/m_s$ and the string coupling g_s. A dual M-theory description is characterized, at least at low energies, by the eleven-dimensional Planck length l_P^{11} and the compactification radius R_{11}. By matching the low energy supergravity theories one finds $R_{11}/l_P^{11} \sim g_s^{2/3}$ [19], while by matching the IIA string tension to the tension of a membrane wrapped around the eleventh dimension one finds $l_s^2 = (l_P^{11})^3/R_{11}$. This fixes the hierarchy of scales

$$\begin{aligned} &l_s \\ l_P^{10} &\sim g_s^{1/4} l_s\,, \\ l_P^{11} &\sim g_s^{1/3} l_s\,, \\ R_{11} &\sim g_s l_s\,, \end{aligned}$$

where we have also listed the ten-dimensional Planck length l_P^{10}.

We now show that zero-brane dynamics, in a regime accurately described by a low energy effective theory, can reliably probe distances much shorter than the string scale. The mass of a D-brane is determined by a BPS formula, $m_{\text{brane}} \sim m_s/g_s$. For zero-branes this matches their interpretation as Kaluza-Klein modes with mass $\sim 1/R_{11}$. Consider scattering two zero-branes at a characteristic velocity $v_{\text{brane}} \sim g_s^\alpha$. We work at weak coupling, so that string loops do not affect the dynamics, and choose an exponent $\alpha > 0$, so that at weak coupling the branes are moving very slowly. The kinetic energy of the branes is then

$$E_{\text{brane}} = \frac{1}{2}\, m_{\text{brane}} v_{\text{brane}}^2 \sim g_s^{2\alpha-1} m_s\,.$$

For $\alpha > 1/2$ the kinetic energy is extremely small, much smaller than the string mass, and we are justified in analyzing the dynamics in a low energy effective theory. In particular, effects due to massive string states (α' corrections to the effective theory) can be neglected. Although the energy is small, the momentum does not have to be small. The momentum

$$p_{\text{brane}} = m_{\text{brane}} v_{\text{brane}} \sim g_s^{\alpha-1} m_s$$

can be much larger than the string scale provided $\alpha < 1$. Thus there is a range of exponents, $1/2 < \alpha < 1$, for which we can use low energy field theory to accurately describe the dynamics of the branes, and in which we can probe distances $\sim 1/p_{\text{brane}}$ all the way down to $g_s^{1/2} l_s$.

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We now formulate the quantum mechanics problem which governs the dynamics of two zero-branes. We take the two zero-branes to have equal Ramond-Ramond U(1) charges, so that when at rest they form a BPS saturated state. (Branes with unequal charges will have radically different dynamics [10].) The relevant degrees of freedom are the massless modes of the open strings which are attached to the brane. These correspond to the dimensional reduction of an $N = 1$ supersymmetric U(2) gauge theory from $9 + 1$ to $0 + 1$ dimensions. Namely, on the world line of the brane we have fields

$$A_0 = \frac{i}{2}(A_0^0 \mathbb{1} + A_0^a \sigma^a),$$
$$\phi_i = \frac{i}{2}(\phi_i^0 \mathbb{1} + \phi_i^a \sigma^a),$$
$$\psi_A = \frac{i}{2}(\psi_A^0 \mathbb{1} + \psi_A^a \sigma^a),$$

where A_0 is a single-component U(2) gauge field, $i = 1, \ldots, 9$ labels the adjoint Higgs fields ϕ_i, $A = 1, \ldots, 16$ labels the real adjoint fermions ψ_A, and $a = 1, 2, 3$ is an SU(2) index.

A generic expectation value for the Higgs fields ϕ_i^a will break the U(2) gauge symmetry down to $U(1) \times U(1)$ at low energy. In terms of zero-branes, this corresponds to the fact that two widely separated zero-branes are described by a $U(1) \times U(1)$ gauge theory, one U(1) factor associated with each brane. The two U(1) Higgs fields in the low energy theory are position coordinates for the branes [1, 2]. [The adjoint representation of U(1) is trivial, so these Higgs fields are gauge invariant. Although one normally expects a gauge field in d dimensions to have $d - 2$ degrees of freedom, upon reduction to $0 + 1$ dimensions an Abelian gauge field has $d - 1$ degrees of freedom. This makes a particle interpretation possible.] This follows from the fact that in the σ model describing an open string attached to the brane

$$\frac{1}{4\pi\alpha'}\int_\Sigma g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu h^{ab} + \oint_{\partial\Sigma}\phi_i \partial_\perp X^i + \cdots$$

the background field $2\pi\alpha'\phi_i$ couples to the momentum conjugate to X^i, and therefore translates the end points of the open string.

The effective action for these fields arises at leading order from an open string disk diagram [2, 21–23]. At low energies, corresponding to weak field strengths, we expect the dynamics to be governed by the standard super Yang-Mills action. In units where $2\pi\alpha' = 1$ the action is

$$S = \int dt\, \frac{1}{2g_s}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} - i\,\mathrm{Tr}\overline{\psi}\Gamma^\mu D_\mu \psi\,.$$

The $1/g_s$ reflects the disk origin of this action; we have absorbed a similar factor in front of the fermion term by rescaling ψ. We have

$$F_{0i} = \partial_0\phi_i + [A_0, \phi_i],$$
$$F_{ij} = [\phi_i, \phi_j],$$
$$D_0\psi = \partial_0\psi + [A_0, \psi],$$
$$D_i\psi = [\phi_i, \psi],$$

and adopt a basis in which

$$\Gamma^0 = \sigma^2 \otimes \mathbb{1}_{16\times 16},$$
$$\Gamma^i = i\sigma^1 \otimes \gamma^i,$$

where γ^i are real symmetric 16×16 matrices given explicitly in [24].

When expanded, the action is a sum of three terms. The first term contains the U(1) gauge degrees of freedom:

$$S_{\mathrm{cm}} = \int dt\, \frac{1}{2g_s}\dot{\phi}_i^0\dot{\phi}_i^0 + \frac{i}{2}\psi^0\dot{\psi}^0.$$

This is an unconstrained system which describes free motion of the center of mass. The second term, involving SU(2) degrees of freedom, governs the relative motion of the two branes:

$$S_{\mathrm{relative}} = \int dt\, \frac{1}{2g_s}\dot{\phi}_i^a\dot{\phi}_i^a + \frac{i}{2}\psi^a\dot{\psi}^a - \frac{1}{4g_s}|\phi_i \times \phi_j|^2 + \frac{i}{2}\epsilon_{abc}\phi_i^a\psi^b\gamma^i\psi^c.$$

In the final term, A_0 appears as a Lagrange multiplier:

$$S_{\mathrm{constraint}} = \int dt\, \frac{1}{g_s}\epsilon_{abc}A_0^a\dot{\phi}_i^b\phi_i^c + \frac{1}{2g_s}|A_0 \times \phi_i|^2 + \frac{i}{2}\epsilon_{abc}A_0^a\psi^b\psi^c.$$

The equations of motion for A_0 require SU(2) invariance, which we impose as a constraint on physical states.

Quantization is straightforward. In terms of eight complex fermions $\chi_M^a = \frac{1}{\sqrt{2}}(\psi_M^a + i\psi_{M+8}^a)$, the Hamiltonian is

$$H = \frac{g_s}{2}|\pi_i|^2 + \frac{1}{4g_s}|\phi_i \times \phi_j|^2 - \sum_{i=1}^{7}\epsilon_{abc}\phi_i^a\overline{\chi}^b\tilde{\gamma}^i\chi^c - \frac{1}{2}\epsilon_{abc}\phi_8^a(\chi^b\chi^c - \overline{\chi}^b\overline{\chi}^c) - \frac{i}{2}\epsilon_{abc}\phi_9^a(\chi^b\chi^c + \overline{\chi}^b\overline{\chi}^c),$$

where $\tilde{\gamma}^i$ are a set of seven real, antisymmetric 8×8 matrices given in [24], and

$$i[\pi_i^a, \phi_j^b] = \delta_{ij}\delta^{ab}, \qquad \{\chi_M^a, \overline{\chi}_N^b\} = \delta_{MN}\delta^{ab}.$$

Without any dynamical analysis, a simple argument shows that the only length scale intrinsic to this problem is the eleven-dimensional Planck length. Recall that ϕ and

π are the distance and momentum measured in string units. Introduce rescaled fields

$$\phi = g_s^{1/3}\phi_{11}, \qquad \pi = g_s^{-1/3}\pi_{11},$$

which are the corresponding quantities measured in eleven-dimensional Planck units. In terms of these fields, the Hamiltonian (still measured in string units) has an overall factor of $g_s^{1/3}$, but no other coupling constant dependence. This shows that the only length scale in the problem is the eleven-dimensional Planck length, and that this length scale can be probed with an energy $\sim g_s^{1/3} m_s$ that is well below the string scale. We therefore expect zero-brane scattering to exhibit some interesting features at a momentum of order the eleven-dimensional Planck scale.

To gain additional insight, we have analyzed a toy problem: s-wave scattering in the $(0 + 1)$-dimensional theory obtained by dimensional reduction of $(2 + 1)$-dimensional $N = 1$ U(2) Yang-Mills theory. The toy problem has two adjoint Higgs fields and a pair of real adjoint fermions. The Hamiltonian for relative motion is

$$\begin{aligned} H = {} & \frac{g_s}{2}(|\pi_1|^2 + |\pi_2|^2) + \frac{1}{2g_s}|\phi_1 \times \phi_2|^2 \\ & - \frac{i}{2}\epsilon_{abc}(\phi_1^a + i\phi_2^a)\chi^b\chi^c \\ & - \frac{i}{2}\epsilon_{abc}(\phi_1^a - i\phi_2^a)\overline{\chi}^b\overline{\chi}^c \end{aligned}$$

with the constraint that physical states must be SU(2) invariant. For a given angular momentum, the bosonic fields can be conveniently characterized by the two invariants

$$r^2 = |\phi_1|^2 + |\phi_2|^2,$$
$$\sin 2\theta = \frac{2|\phi_1 \times \phi_2|}{|\phi_1|^2 + |\phi_2|^2}.$$

The angle θ ranges from 0 to $\pi/4$. When $\theta = 0$ the two SU(2) vectors are aligned, in which case r can be interpreted as the distance between the branes.

We diagonalize the total angular momentum $j = l + s$, where l is the orbital angular momentum and s is the spin of the branes. The Hamiltonian for a given j acts on a four-dimensional space of fermion states. Written in a convenient basis, the Hamiltonian for s-wave ($j = 0$) scattering is

$$H = \begin{pmatrix} h_{l=0} & -\frac{i}{\sqrt{2}}re^{i\theta} & \frac{i}{\sqrt{2}}re^{-i\theta} & 0 \\ \frac{i}{\sqrt{2}}re^{-i\theta} & h_{l=1} & 0 & 0 \\ -\frac{i}{\sqrt{2}}re^{i\theta} & 0 & h_{l=1} & 0 \\ 0 & 0 & 0 & h_{l=1} \end{pmatrix},$$

where the bosonic part of the Hamiltonian is

$$\begin{aligned} h_l = {} & -\frac{g_s}{2}\left(\frac{1}{r^5}\partial_r r^5 \partial_r + \frac{1}{r^2 \sin 4\theta}\partial_\theta \sin 4\theta \partial_\theta\right) \\ & + \frac{g_s}{2}\frac{l^2}{r^2\cos^2 2\theta} + \frac{1}{8g_s}r^4 \sin^2 2\theta. \end{aligned}$$

Note that h_l is self-adjoint in the integration measure $\int r^5 \sin 4\theta \, dr \, d\theta$.

There are low energy states, localized at large r and small θ, which describe widely separated branes. These states exist because supersymmetry is unbroken, so the zero point energies cancel. A basis for these states is given by the wave functions

$$\Psi_k = \frac{1}{r}e^{-ikr}e^{-r^3\theta^2/2g_s}u_0,$$

$$u_0 = \begin{pmatrix} 1/\sqrt{2} \\ -i/2 \\ i/2 \\ 0 \end{pmatrix}.$$

Note that Ψ_k becomes an exact eigenfunction as $r \to \infty$, $\theta \to 0$, with $r^{3/2}\theta$ held fixed. (For a fixed distance r between the branes, Ψ_k spreads a distance of order $g_s^{1/2}$ in the $y = r\theta$ direction. The physical meaning of this length scale is not obvious, as y is a noncommuting coordinate. We are grateful to Steve Shenker for pointing this phenomenon out to us.) It represents an incoming s-wave scattering state with relative momentum k and energy $\frac{1}{2}g_s k^2$.

For small r, the potential energy and also the off-diagonal terms in the Hamiltonian are negligible, and the quantum mechanics problem reduces to free motion. This suggests that an incoming s-wave scattering state could get trapped near $r = 0$, and resonate there for some time before escaping back to infinity. This phenomenon has the D-brane interpretation that s-wave scattering of two D-branes can produce a resonant state, in which the two D-branes are bound together by a condensate of open strings.

We can make the resonant behavior manifest with the following crude calculation. To find the wave function of the resonance, we modify the Hamiltonian in a way which preserves its form at small r, but which lifts the supersymmetric ground state. A suitable modification is to change the potential energy in the Hamiltonian, replacing $\frac{1}{8g_s}r^4\sin^2 2\theta u_0 u_0^\dagger$ with $\frac{1}{8g_s}r^4 u_0 u_0^\dagger$. This modification prevents the scattering states Ψ_k (which are proportional to u_0) from escaping to $r = \infty$, without affecting the states orthogonal to u_0. It turns the resonance into a genuine eigenstate localized near $r = 0$, and we can estimate its wave function. The lowest-lying resonance has a wave function

$$\Psi_{\rm res} \approx e^{-r^3/6g_s}\begin{pmatrix} 1 \\ Ag_s^{-1/3}r\sqrt{\cos 2\theta} \\ A^* g_s^{-1/3}r\sqrt{\cos 2\theta} \\ 0 \end{pmatrix}$$

with an energy $E_{\text{res}} \approx 1.7 g_s^{1/3} m_s$ and $A \approx -0.23 - i0.28$ estimated by variational methods. Note that the resonance indeed has a size of order the eleven-dimensional Planck length $l_P^{11} = g_s^{1/3} l_s$.

By matching this resonance wave function onto a superposition of scattering wave functions at r of order $g_s^{1/3}$, we can obtain a crude estimate of the s-wave phase shift near resonance:

$$\delta_0(k) \approx -kl_P^{11} - \frac{\pi}{4} - \tan^{-1} ckl_P^{11},$$

where c is a number of order 1. This phase shift exhibits a resonant feature at a momentum k of order $1/l_P^{11}$.

Many of the features of the toy model should survive in the full $(9+1)$-dimensional problem. We expect that a rich spectrum of resonances should exist, with a characteristic size set by the eleven-dimensional Planck length. But new phenomena should also arise, including the bound states at threshold which are predicted by duality.

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